OU-HET 259
hep-th 9703***,
March, 1997

Schwinger-Dyson Equation for Supersymmetric Yang-Mills Theory: Manifestly Supersymmetric Form [1]

H. Itoyama
and
H. Takashino

Department of Physics,
Graduate School of Science, Osaka University,
Toyonaka, Osaka, 560 Japan

Abstract

We study our Schwinger-Dyson equation as well as the large N_c loop equation for supersymmetric Yang-Mills theory in four dimensions by the $N = 1$ superspace Wilson-loop variable. We are successful in deriving a new manifestly supersymmetric form in which a loop splitting and joining are represented by a manifestly supersymmetric as well as supergauge invariant operation in superspace. This is found to be a natural extension from the abelian case. We solve the equation to leading order in perturbation theory or equivalently in the linearized approximation, obtaining a desirable nontrivial answer. The super Wilson-loop variable can be represented as the system of one-dimensional fermion along the loop coupled minimally to the original theory. One-loop renormalization of the one-point Wilson-loop average is explicitly carried out, exploiting this property. The picture of string dynamics obtained is briefly discussed.

[1] This work is supported in part by Grant-in-Aid for Scientific Research (07640403) from the Ministry of Education, Science and Culture, Japan.

I. Introduction

Intensive efforts have recently been directed to both supersymmetric gauge theories and superstring unification. Main motivation is that we must be able to discuss nonperturbative phenomena for these intelligibly before we confront ourselves to reality. In one side we have rich phenomena which supersymmetric gauge theories offer. These will become hints to how string theory should be formulated in the other. Connection between superstrings and gauge theories is thus beginning to play important roles in recent activities and will continue to progress in various directions. It has been known, in a related context, that the formulation of gauge theories as Schwinger-Dyson equations which exploit Wilson-loops as basic variables provides a natural framework to discuss nonperturbative phenomena such as confinement and dynamical symmetry breaking. At the same time, it exhibits string interactions contained in gauge theories as loop dynamics [1, 2, 3]. We wish to provide a significant step to this approach in this paper, which is relevant to current issues.

In the previous paper [4], we initiated the (super)-gauge invariant formulation of supersymmetric gauge theories as Schwinger-Dyson equations. We proposed to use the Wilson-loop having the connection one-form in $N = 1$ superspace as fundamental variable. We will refer to this variable as super Wilson-loop [5]-[7] in this paper. This setup owes some of the good properties to the geometric formulation [8, 9] of supersymmetric gauge theories on $N = 1$ superspace, which is by now well understood in the literature [2].

We have been successful in deriving the Schwinger-Dyson equation as well as the large N_c loop equation in a closed form consisting of the super Wilson-loop alone. The (super)-area derivative acting on the super Wilson-loop, which includes the Grassmannian directions, is found to be a central geometric operation to our equation. The final form of our equation looked, however, complicated. Although it is closed with respect to the super Wilson-loop, it was not clear how we could make further progress with this. The basic reason for this complexity comes from the fact that we have to work with the Wess-Zumino gauge in order to carry out algebras in intermediate steps. Manifest supersymmetry is lost this way. One of the upshots of the present paper is that we are able to overcome this difficulty and to put the equation in the manifestly supergauge invariant form.

Another aspect of the super Wilson-loop is concerned with the question of the removal of infinities. The super Wilson-loop is a composite operator and contains infinite number of insertions of gauginos and the auxiliary D field into the ordinary Wilson-loop along the curve. One may think that renormalization of this composite operator is prohibitively complicated.

[2] There were several attempts in the earlier literature [10, 11], trying to derive supersymmetric loop equations. These were hampered by the lack of this geometrical formulation.

We will show that this is not the case. The renormalization of the super Wilson-loop is no more complicated than that of the ordinary Wilson-loop [12]-[17].

In the next section, we introduce the superspace one-form from the vector superfield V which is followed by the construction of the super Wilson-loop. We list several important properties of the super-Wilson loop. In section three, we first derive the Schwinger-Dyson equation in the abelian case. Although this is a free theory, it provides us with an important insight into how we "gauge-unfix" the equation in subsequent sections. In section four, we present the derivation of our equation in the Wess-Zumino gauge. The equation we obtained is (4.36)

$$
\begin{aligned}
& \frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^{\alpha}\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}\langle trW_S[C]\rangle \\
= \; & -2\oint\{-\frac{i}{2}dy^a(\eta-\theta)\sigma_a(\bar{\eta}-\bar{\theta})-\frac{1}{2}dy^a\delta(\eta-\theta)\bar{\theta}\bar{\sigma}^b\sigma_a\bar{\eta}\partial_b \\
& \quad +\bar{\eta}^2\left(d\eta(\eta-\theta)+id\eta\sigma^a\bar{\theta}\delta(\eta-\theta)\partial_a\right)\}\langle tr\,W_S[C_{z'z}]\delta^4(y-y')tr\,W_S[C_{zz'}]\rangle \\
& +2\oint\frac{1}{8}\bar{\eta}^2\delta(\eta-\theta)\delta^4(y-y')dy^a(\bar{\sigma}^b\sigma_a\bar{\theta})^{\dot{\alpha}} \\
& \times\left\{\langle(\frac{\delta}{\delta\Sigma^{b\dot{\alpha}}(z)}tr\,W_S[C_{z'z}])tr\,W_S[C_{zz'}]\rangle-\langle tr\,W_S[C_{z'z}](\frac{\delta}{\delta\Sigma^{b\dot{\alpha}}(z)}tr\,W_S[C_{zz'}])\rangle\right\}.
\end{aligned}
\tag{4.36}
$$

Only the ordinary gauge invariance is kept intact.

Section five deals with our advancement to the manifestly supersymmetric form. Being motivated by the abelian case, we are able to find how to make (4.36) into manifestly supergauge invariant. This amounts to recovering the Wess-Zumino gauge volume, which is thrown away upon partial gauge fixing. The final form of our equation reads

$$
\begin{aligned}
& \frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^{\alpha}\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}\langle tr\,W_S[C]\rangle \\
= \; & \oint\mathcal{D}_f^{\alpha}\mathrm{D}_{\alpha}\delta(z-z')\langle tr\,W_S[C_{z'z}]tr\,W_S[C_{zz'}]\rangle\,, \\
& \mathcal{D}_f^{\alpha}=-e^a\frac{i}{4}\bar{\sigma}_a^{\dot{\alpha}\alpha}\bar{\mathrm{D}}_{\eta\dot{\alpha}}+e^{\alpha}\,, \\
& \langle\cdots\rangle=\int[d\mathrm{V}]\,\mathrm{e}^{i\mathrm{S}_{SYM}}\,\cdots\,.
\end{aligned}
\tag{5.13}
$$

We briefly discuss the picture of string dynamics obtained from the sequence of the Schwinger-Dyson equations beginning with this one. In section six, we solve the equation in the abelian case. This is equivalent to solving the full-fledged nonabelian case in the linearized approximation or to the leading order in perturbation theory. In section seven, we deal with the problem of removing infinities and renormalization. Representing the super Wilson-loop as

the first quantized system of one-dimensional fermion along the loop, this problem becomes that for $\mathcal{L}_{SYM} + \mathcal{L}_{path}$. This can then be handled in the standard fashion. We carry out the one-loop renormalization explicitly and the infinities are shown to cancel by local counterterms. In the final section, we discuss on a few points. In the first two of three appendix, we give some pedagogical details on the area derivative and the Migdal-Makeenko equation. In the final appendix, we summarize the basic formulas on the connection one-form in superspace and the vector superfield known in the literature and used frequently in the text. Unless written explicitly, we follow the notation of [8].

II. Supersymmetric Wilson-loop

Let us consider the supersymmetric Schwinger-Dyson equation using the Wilson-loop variable. In parallel to the ordinary path ordered exponential of the one-form $dx^a v_a$ on spacetime x, we define the path ordered exponential of the one-form $\mathcal{A}$ on superspace z [5]-[7], [10]. From (3.77) and (C.23), $\mathcal{A}$ is written in terms of the vector superfield V as

$$\begin{aligned}
\mathcal{A} = e^{\mathrm{A}}\mathcal{A}_{\mathrm{A}} &= (dx^a - id\eta\sigma^a\bar{\eta} + i\eta\sigma^a d\bar{\eta})\mathcal{A}_a + d\eta^\alpha\mathcal{A}_\alpha\,, \\
\mathcal{A}_\alpha &= -\mathrm{e}^{-\mathrm{V}}\mathrm{D}_\alpha\mathrm{e}^{\mathrm{V}}\,, \\
\bar{\mathcal{A}}_{\dot{\alpha}} &= 0\,, \\
\mathcal{A}_a &= \frac{i}{4}\bar{\sigma}_a^{\dot{\alpha}\alpha}\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{e}^{-\mathrm{V}}\mathrm{D}_\alpha\mathrm{e}^{\mathrm{V}}\,,
\end{aligned} \tag{2.1}$$

where $(x, \eta, \bar{\eta})$ denotes the superspace coordinates. In terms of $(y = x + i\eta\sigma^a\bar{\eta}, \eta, \bar{\eta})$, (2.1) becomes

$$\mathcal{A} = (dy^a - 2id\eta\sigma^a\bar{\eta})\mathcal{A}_a + d\eta^\alpha\mathcal{A}_\alpha\,. \tag{2.2}$$

We introduce the curve $C_{z_1 z_2}$ on superspace, using one dimensional parameter t:

$$\begin{aligned}
C_{z_1 z_2} : z^M &= z^M(t) \\
&= (x(t), \eta(t), \bar{\eta}(t)), \\
0 \leq t \leq 1, &\qquad z(0) = z_2, \quad z(1) = z_1.
\end{aligned} \tag{2.3}$$

One can think of the Grassmannian coordinates $\eta(t)$ or $\bar{\eta}(t)$ as being defined by the infinite number of Grassmannian parameters via Taylor expansion

$$\begin{aligned}
\eta(t) &= \sum_{n=0}^{\infty}\frac{1}{n!}\eta^{(n)}t^n\,, \\
\bar{\eta}(t) &= \sum_{n=0}^{\infty}\frac{1}{n!}\bar{\eta}^{(n)}t^n\,.
\end{aligned}$$

Supersymmetric path-ordered exponential $W_S[C_{z_1z_2}]$ is given by

$$
\begin{aligned}
W_S[C_{z_1z_2}] &= \sum_{n=0}^{\infty}\int_0^1 \mathcal{A}(z(t_1))\int_0^{t_1}\mathcal{A}(z(t_2))\cdots\int_0^{t_{n-1}}\mathcal{A}(z(t_n)) \\
&\equiv P\exp\int_{z_2}^{z_1}\mathcal{A}(z)\ , \qquad (2.4) \\
\mathcal{A}(z(t)) &= dt\frac{dz^M}{dt}\mathcal{A}_M(z(t)) \\
&= dt\{\frac{dx^a}{dt} - i\frac{d\eta}{dt}\sigma^a\bar{\eta} + i\eta\sigma^a\frac{d\bar{\eta}}{dt}\}\mathcal{A}_a \\
&\quad +dt\frac{d\eta^\alpha}{dt}\mathcal{A}_\alpha\,. \qquad (2.5)
\end{aligned}
$$

In terms of $(y,\eta,\bar{\eta})$

$$
\mathcal{A}(z(t)) = dt\{\frac{dy^a}{dt} - 2i\frac{d\eta}{dt}\sigma^a\bar{\eta}\}\mathcal{A}_a + dt\frac{d\eta^\alpha}{dt}\mathcal{A}_\alpha\,. \qquad (2.6)
$$

Here the capital P denotes the path-ordered product in superspace. Since the lowest component of $\mathcal{A}_a$ is $-\frac{i}{2}v_a$, W_S becomes the ordinary path-ordered exponential W if we take $\eta=\bar{\eta}=0$.

The $W_S[C_{z_1z_2}]$ can be characterized by

$$
\begin{aligned}
\frac{d\phi(z(t))}{dt} &= \mathcal{A}(z(t))\phi(z(t))\,, \\
\phi(z_1) &= W_S[C_{z_1z_2}]\phi(z_2)\,, \qquad (2.7)
\end{aligned}
$$

where $\phi(z)$ denotes a certain superfield belonging to a representation of the gauge group. Note that the one-form $\mathcal{A}$ is given by (2.1) and is expressed with the vector superfield V defined by (C.24). It contains not only the usual Yang-Mills fields v_a but also its partners $\lambda,\bar{\lambda}$ and D. From the definition of $W_S[C_{z_1z_2}]$, namely (2.4), we see that $W_S[C_{z_1z_2}]$ has infinite number of insertions $\lambda,\bar{\lambda}$ and D along the curve $C_{z_1z_2}$. The fields $\lambda,\bar{\lambda}$ and D all belong to the adjoint representation and should be treated in the same way as v_a. The $W_S[C_{z_1z_2}]$ is natural from that perspective, which becomes most evident in the large N_c limit. The operator W_S possesses the properties similar to those of the ordinary path-ordered exponential W.

(A) W_S has reparametrization invariance for curves in superspace.

(B) For the addition of two curves, $W_S[C_{z_1z_2}] = W_S[C_{z_1z}]W_S[C_{zz_2}] = W_S[C_{z_1z}+C_{zz_1}]$.

(C) For the deformation of the curve $C_{z_1z_2}$:$z(t)\to z(t)+\delta z(t)$, we obtain

$$
\begin{aligned}
\delta W_S[C_{z_1z_2}] &= \int_{z_2}^{z_1} dz^M\delta z^N W_S[C_{z_1z}]\mathcal{F}_{NM}(z)W_S[C_{zz_2}] \\
&\quad +\{\delta z_1^M\mathcal{A}_M(z_1)W_S[C_{z_1z_2}] - W_S[C_{z_1z_2}]\delta z_2^M\mathcal{A}_M\} \qquad (2.8)
\end{aligned}
$$

where $\delta z_1 = \delta z(1),\ \delta z_2 = \delta z(0)$ and $\mathcal{F}_{NM}(z)$ having Einstein indices are the components of the field strength $\mathcal{F}$

$$\begin{aligned}\mathcal{F} &= d\mathcal{A} + \mathcal{A}\mathcal{A} = \frac{1}{2} dz^M dz^N \mathcal{F}_{NM} \\ &= \frac{1}{2} dz^M dz^N \{\partial_N \mathcal{A}_M - (-)^{|N||M|} \partial_M \mathcal{A}_N \\ &\qquad\qquad -\mathcal{A}_N \mathcal{A}_M + (-)^{|N||M|} \mathcal{A}_M \mathcal{A}_N\}\,. \end{aligned} \tag{2.9}$$

From (2.8), we can define the end point derivatives of $W_S[C_{z_1 z_2}]$:

$$\frac{\partial}{\partial z_1^M} W_S[C_{z_1 z_2}] = \mathcal{A}_M(z_1) W_S[C_{z_1 z_2}] \ \ , \ \ \frac{\partial}{\partial z_2^M} W_S[C_{z_1 z_2}] = -W_S[C_{z_1 z_2}] \mathcal{A}_M(z_2)\,. \tag{2.10}$$

Hitting with the derivative D_A having flat indices (see (C.6)), we obtain

$$\begin{aligned}\mathrm{D}_A^{(z_1)} W_S[C_{z_1 z_2}] &= \mathcal{A}_A(z_1) W_S[C_{z_1 z_2}]\,, \\ \mathrm{D}_A^{(z_2)} W_S[C_{z_1 z_2}] &= -W_S[C_{z_1 z_2}] \mathcal{A}_A(z_2)\,. \end{aligned} \tag{2.11}$$

By virtue of the condition $\mathcal{A}_{\dot\alpha} = 0$, $W_S[C_{z_1 z_2}]$ satisfies

$$\mathrm{D}_{\dot\alpha} W_S[C_{z_1 z_2}] = 0\,. \tag{2.12}$$

(D) We can act the area derivative in superspace on W_S in a similar way to the bosonic case. We find

$$\begin{aligned}\frac{\delta}{\delta \Sigma^{NM}(z_1)} W_S[C_{z_1 z_2}] &= \mathcal{F}_{NM}(z_1) W_S[C_{z_1 z_2}]\,, \\ \delta \Sigma^{NM} &= -\frac{1}{2} \delta s \delta t \{\frac{\partial u^M}{\partial s}\frac{\partial u^N}{\partial t} - (-)^{|M||N|} \frac{\partial u^N}{\partial s}\frac{\partial u^M}{\partial t}\}\,. \end{aligned} \tag{2.13}$$

From the definition for the field strength $\mathcal{F}$,

$$\mathcal{F} = \frac{1}{2} dz^M dz^N \mathcal{F}_{NM} = \frac{1}{2} e^B e^A \mathcal{F}_{AB}\,, \tag{2.14}$$

we may relate the components of $\mathcal{F}_{AB}$ with those of $\mathcal{F}_{NM}$

$$\mathcal{F}_{AB} = -(-)^{|B||M|} e_A{}^M e_B{}^N \mathcal{F}_{NM}\,, \tag{2.15}$$

where the matrix $e_A{}^M$ is defined by (C.10). Defining the area derivative $\frac{\delta}{\delta\Sigma^{AB}}$ having flat indices by

$$\frac{\delta}{\delta \Sigma^{AB}} \equiv -(-)^{|B||M|} e_A{}^M e_B{}^N \frac{\delta}{\delta \Sigma^{NM}}\,, \tag{2.16}$$

we rewrite (2.13) with respect to the flat indices:

$$\frac{\delta}{\delta\Sigma^{AB}(z_1)} W_S[C_{z_1z_2}] \quad = \quad \mathcal{F}_{AB}(z_1) W_S[C_{z_1z_2}] \,. \tag{2.17}$$

As we impose the flatness condition (C.22) on the one-form $\mathcal{A}$, we have

$$\frac{\delta}{\delta\Sigma^{\alpha\beta}} W_S = \frac{\delta}{\delta\Sigma^{\dot\alpha\dot\beta}} W_S = \frac{\delta}{\delta\Sigma^{\alpha\dot\alpha}} W_S = 0 \,. \tag{2.18}$$

(E) The supergauge transformations of the components $\mathcal{A}_M$ of $\mathcal{A}$ are

$$\mathcal{A}_M \longrightarrow \mathcal{A}'_M = -X^{-1}\partial_M X + X^{-1}\mathcal{A}_M X \,,$$

where X is an element of the super gauge group. The $\phi(z)$ in (2.7) transforms as

$$\phi(z) \longrightarrow \phi'(z) = X^{-1}\phi(z) \,. \tag{2.19}$$

From these transformation laws, we obtain

$$W_S[C_{z_1z_2}] \longrightarrow W'_S[C_{z_1z_2}] = X^{-1}(z_1) W_S[C_{z_1z_2}] X(z_2) \,. \tag{2.20}$$

We now take the special solution to the flatness condition (C.22) and look at the components of $\mathcal{A}$ having flat indices given by (2.1). The supergauge transformations which keep this condition are more restricted than the most general supergauge transformations as is explained in Appendix C. The transformation laws of W_S

$$W_S[C_{z_1z_2}] \longrightarrow W'_S[C_{z_1z_2}] = \mathrm{e}^{-\Lambda(z_1)} W_S[C_{z_1z_2}] \mathrm{e}^{\Lambda(z_2)} \,, \tag{2.21}$$

where $\Lambda(z)$ is a chiral superfield. The above transformation is consistent with (2.12) as

$$\begin{aligned} \mathrm{D}_{\dot\alpha} W'_S[C_{z_1z_2}] \quad &= \quad \mathrm{e}^{-\Lambda(z_1)} \mathrm{D}_{\dot\alpha} W_S[C_{z_1z_2}] \mathrm{e}^{\Lambda(z_2)} \\ &= \quad 0 \,. \end{aligned}$$

It is clear that, in the case that the curve $C_{z_1z_2}$ is a closed loop C_{zz} in superspace, $trW_S[C_{zz}]$ is invariant under $\mathrm{e}^{\mathrm{V}} \rightarrow \mathrm{e}^{\mathrm{V}'} = \mathrm{e}^{\Lambda^\dagger}\mathrm{e}^{\mathrm{V}}\mathrm{e}^{\Lambda}$.

III. The Schwinger-Dyson equation in the abelian Case

As is well-known, the supersymmetric pure Yang-Mills Lagrangian reads as

$$\begin{aligned} \mathcal{L}_{SYM} \quad &= \quad \frac{1}{8g^2} tr(\mathcal{W}\mathcal{W}) \mid_{\theta\theta \text{ component}} \\ &= \quad \frac{1}{8g^2}\int d^2\theta tr(\mathcal{W}\mathcal{W}) = \frac{1}{8g^2}\int d^2\theta d^2\bar\theta \, \delta(\bar\theta) tr(\mathcal{W}\mathcal{W}) \,. \end{aligned} \tag{3.1}$$

Furthermore

$$
\begin{aligned}
\mathcal{L}_{SYM} &= \frac{1}{8g^2}\int d^2\theta d^2\bar{\theta}\,\delta(\bar{\theta})tr\{-\frac{1}{4}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{e}^{-\mathrm{V}}\mathrm{D}^\alpha\mathrm{e}^{\mathrm{V}}\mathcal{W}_\alpha\} \\
&= -\frac{1}{32g^2}\int d^2\theta d^2\bar{\theta}\,(\bar{\mathrm{D}}\bar{\mathrm{D}}\delta(\bar{\theta}))tr\{\mathrm{e}^{-\mathrm{V}}\mathrm{D}^\alpha\mathrm{e}^{\mathrm{V}}\mathcal{W}_\alpha\} \\
&= -\frac{1}{8g^2}\int d^2\theta d^2\bar{\theta}\,tr\{\mathcal{A}^\alpha\mathcal{W}_\alpha\}\,, \qquad (3.2)
\end{aligned}
$$

where we have used $\bar{\mathrm{D}}_{\dot{\alpha}}\mathcal{W}_\alpha = 0$ and omitted spacetime total derivatives.

Let us define the functional derivative of the vector superfield V by

$$
\begin{aligned}
\frac{\delta}{\delta\mathrm{V}(z)} &= \theta^2\bar{\theta}^2\{\frac{\delta}{\delta\mathrm{C}(x)} - \frac{1}{2}\Box\frac{\delta}{\delta\mathrm{D}(x)}\} \\
&+\frac{2}{i}\theta^2\bar{\theta}_{\dot{\beta}}\{\frac{\delta}{\delta\bar{\chi}^{\dot{\beta}}(x)} - \frac{i}{2}\partial_a\frac{\delta}{\delta\lambda^\alpha(x)}\bar{\sigma}^{a\dot{\beta}\alpha}\} - \frac{2}{i}\bar{\theta}^2\theta^\beta\{\frac{\delta}{\delta\chi^\beta(x)} - \frac{i}{2}\partial_a\frac{\delta}{\delta\bar{\lambda}_{\dot{\alpha}}(x)}\sigma^a_{\beta\dot{\alpha}}\} \\
&+\frac{2}{i}\bar{\theta}^2\frac{\delta}{\delta(\mathrm{M}(x)+i\mathrm{N}(x))} - \frac{2}{i}\theta^2\frac{\delta}{\delta(\mathrm{M}(x)-i\mathrm{N}(x))} \\
&+2\theta\sigma^a\bar{\theta}\frac{\delta}{\delta v_a(x)} - \frac{2}{i}\bar{\theta}_{\dot{\alpha}}\frac{\delta}{\delta\bar{\lambda}_{\dot{\alpha}}(x)} + \frac{2}{i}\theta^\alpha\frac{\delta}{\delta\lambda^\alpha(x)} + 2\frac{\delta}{\delta\mathrm{D}(x)}\,. \qquad (3.3)
\end{aligned}
$$

In terms of $(y, \theta, \bar{\theta})$,

$$
\begin{aligned}
\frac{\delta}{\delta\mathrm{V}(z)} &= \theta^2\bar{\theta}^2\frac{\delta}{\delta\mathrm{C}(y)} - 2i\theta^2\bar{\theta}_{\dot{\beta}}\frac{\delta}{\delta\bar{\chi}_{\dot{\beta}}(y)} + 2i\bar{\theta}^2\theta^\beta\frac{\delta}{\delta\chi^\beta(y)} - 2i\bar{\theta}^2\frac{\delta}{\delta(\mathrm{M}(y)+i\mathrm{N}(y))} \\
&+2i\theta^2\frac{\delta}{\delta(\mathrm{M}(y)-i\mathrm{N}(y))} + 2\theta\sigma^a\bar{\theta}\frac{\delta}{\delta v_a(y)} + i\theta^2\bar{\theta}^2\partial_a\frac{\delta}{\delta v_a(y)} + 2i\bar{\theta}_{\dot{\alpha}}\frac{\delta}{\delta\bar{\lambda}_{\dot{\alpha}}(y)} \\
&-2i\theta^\alpha\{\frac{\delta}{\delta\lambda^\alpha(y)} - 2i\theta\sigma^a\bar{\theta}\partial_a\frac{\delta}{\delta\lambda^\alpha(y)}\} \\
&+2\{\frac{\delta}{\delta\mathrm{D}(y)} - i\theta\sigma^a\bar{\theta}\partial_a\frac{\delta}{\delta\mathrm{D}(y)}\}\,. \qquad (3.4)
\end{aligned}
$$

These derivatives satisfy

$$
\begin{aligned}
\frac{\delta}{\delta\mathrm{V}(z)}\mathrm{V}(z') &= \delta^4(x-x')\delta(\theta-\theta')\delta(\bar{\theta}-\bar{\theta}') \\
&= \delta^4(y-y')\delta(\theta-\theta')\delta(\bar{\theta}-\bar{\theta}') = \delta(z-z')\,, \qquad (3.5)
\end{aligned}
$$

where $y^a = x^a + i\theta\sigma^a\bar{\theta}$, $y'^a = x'^a + i\eta\sigma^a\bar{\eta}$.

We are now ready to obtain an equation for the abelian case. The field V includes the usual gauge field v_a and their superpartners, gauginos λ, $\bar{\lambda}$ linearly. As in (B.1) we start out

from

$$0=\int[d\mathrm{V}]\frac{\delta}{\delta\mathrm{V}(z')}\{\mathrm{e}^{i\mathrm{S}_{SA}}W_S[C]\}\,, \tag{3.6}$$

$$\mathrm{S}_{SA}=\frac{1}{8g^2}\int d^4xd^2\theta d^2\bar{\theta}\mathcal{A}^\alpha\mathcal{W}_\alpha\,, \tag{3.7}$$

$$W_S[C]=P\exp\oint_C\mathcal{A}=\exp\oint_C\mathcal{A}\,. \tag{3.8}$$

Note that, in the abelian case, $\mathcal{A}$ as well as $\mathcal{W}_\alpha$ is linear in the vector superfield V,

$$\begin{aligned}\mathcal{A}_\alpha &= -\mathrm{D}_\alpha\mathrm{V}\,,\quad \mathcal{A}_{\dot{\alpha}}=0\\ \mathcal{A}_a &= \frac{i}{4}\bar{\mathrm{D}}\bar{\sigma}^a\mathrm{D}\mathrm{V}\,,\\ \mathcal{W}_\alpha &= -\frac{1}{4}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{D}_\alpha\mathrm{V}\,.\end{aligned} \tag{3.9}$$

We find

$$\begin{aligned}\frac{\delta}{\delta\mathrm{V}(z')}\mathrm{e}^{i\mathrm{S}_{SA}} &= i\frac{1}{32g^2}\mathrm{e}^{i\mathrm{S}_{SA}}\frac{\delta}{\delta\mathrm{V}(z')}\int d^4xd^2\theta d^2\bar{\theta}\mathrm{D}^\alpha\mathrm{V}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{D}_\alpha\mathrm{V}\\ &= i\frac{1}{16g^2}\mathrm{e}^{i\mathrm{S}_{SA}}\int d^4xd^2\theta d^2\bar{\theta}(\mathrm{D}^\alpha\delta(z-z'))\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{D}_\alpha\mathrm{V}\\ &= -i\frac{1}{4g^2}\mathrm{D}^\alpha\mathcal{W}_\alpha(z')\mathrm{e}^{i\mathrm{S}_{SA}}\,.\end{aligned} \tag{3.10}$$

The components of $\mathrm{D}^\alpha\mathcal{W}_\alpha=\mathrm{D}\mathcal{W}=0$ correspond to field equations for the fields $(\lambda^\alpha,\lambda_{\dot{\alpha}},v_a,\mathrm{D})$ in (C.24),

$$\begin{aligned}\mathrm{D}\mathcal{W} &= -2D(y)+2\theta\sigma^a\partial_a\bar{\lambda}(y)-2\bar{\theta}\bar{\sigma}^a\partial_a\lambda(y)\\ &\quad -2\theta\sigma_a\bar{\theta}\partial_b v^{ba}(y)+2i\theta\theta\bar{\theta}\Box\bar{\lambda}(y)+2i\theta\sigma^a\bar{\theta}\partial_a\mathrm{D}(y)\\ &= 0\\ &\Longleftrightarrow\ \left\{\begin{array}{r}\mathrm{D}=0\ ,\\ \sigma^a\partial_a\bar{\lambda}=0\ ,\bar{\sigma}^a\partial_a\lambda=0\ ,\\ \partial_a v^{ab}=0\ .\end{array}\right.\end{aligned} \tag{3.11}$$

From the relation $\mathcal{F}_{a\dot{\alpha}}=i\sigma_{a\alpha\dot{\alpha}}\mathcal{W}^{\dot{\alpha}}/2$, we may rewrite (3.10) as

$$\frac{\delta}{\delta\mathrm{V}(z')}\mathrm{e}^{i\mathrm{S}_{SA}}=\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\{\mathrm{D}^\alpha\mathcal{F}_{a\dot{\alpha}}(z')\}\mathrm{e}^{i\mathrm{S}_{SA}}\,. \tag{3.12}$$

Remember that the field strength $\mathcal{F}_{a\dot{\alpha}}$ is obtained by the area derivative on $W_S[C]$. Using this fact and (3.10), we find

$$\{\frac{\delta}{\delta\mathrm{V}(z')}\mathrm{e}^{i\mathrm{S}_{SA}}\}W_S[C]=\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\{\mathrm{D}^\alpha\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}W_S[C]\}\mathrm{e}^{i\mathrm{S}_{SA}}\,. \tag{3.13}$$

On the other hand,

$$\begin{aligned}
\frac{\delta}{\delta \mathrm{V}(z')} W_S[C] &= \oint_C \{\frac{\delta}{\delta \mathrm{V}(z')} \mathcal{A}(z(t))\} W_S[C] \\
&= W_S[C] \oint_C \{\frac{i}{4}(dx^a - id\eta\sigma^a\bar{\eta} + i\eta\sigma^a d\bar{\eta})\bar{\mathrm{D}}_\eta \bar{\sigma}_a \mathrm{D}_\eta \\
&\qquad -d\eta \mathrm{D}_\eta\}\delta(z - z')\,. \qquad (3.14)
\end{aligned}$$

In terms of $(y, \theta, \bar{\theta})$,

$$\begin{aligned}
\frac{\delta}{\delta \mathrm{V}(z')} W_S[C] &= W_S[C] \oint_C \{\frac{i}{4}(dy^a - 2id\eta\sigma^a\bar{\eta})\bar{\mathrm{D}}_\eta \bar{\sigma}_a \mathrm{D}_\eta \\
&\qquad -d\eta \mathrm{D}_\eta\}\delta(z - z')\,, \qquad (3.15)
\end{aligned}$$

where $z = (x, \eta, \bar{\eta})$, $z' = (x', \theta, \bar{\theta})$. We denote by D_η, $\bar{\mathrm{D}}_\eta$ the superderivatives:

$$\begin{aligned}
\mathrm{D}_{\eta\alpha} &= \frac{\partial}{\partial \eta^\alpha} + i(\sigma^a\bar{\eta})_\alpha \frac{\partial}{\partial x'^a}\,, \\
\bar{\mathrm{D}}_{\eta\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\eta}^\alpha} - i(\eta\sigma^a)_{\dot{\alpha}} \frac{\partial}{\partial x'^a}\,. \qquad (3.16)
\end{aligned}$$

Considering both (3.13) and (3.14), we obtain the supersymmetric Schwinger-Dyson equation in the abeian case:

$$\begin{aligned}
\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^\alpha \frac{\delta}{\delta \Sigma^{a\dot{\alpha}}(z')} \mathbf{W}_S[C] = \mathbf{W}_S[C] \oint_C \mathcal{D}_f^\alpha \mathrm{D}_{\eta\alpha}\delta(z - z')\,, \\
\mathbf{W}_S[C] \equiv \langle W_S[C] \rangle = \int [d\mathrm{V}] \mathrm{e}^{i\mathrm{S}_{SA}} W_S[C]\,, \qquad (3.17)
\end{aligned}$$

where we have introduced the derivative $\mathcal{D}_f^\alpha$ by

$$\begin{aligned}
\mathcal{D}_f^\alpha &= -\frac{i}{4}(dx^a - id\eta\sigma^a\bar{\eta} + i\eta\sigma^a d\bar{\eta})\bar{\sigma}_a^{\dot{\alpha}\alpha}\bar{\mathrm{D}}_{\eta\dot{\alpha}} \\
&\quad + d\eta^\alpha \\
&= -\frac{i}{4}(dy^a - 2id\eta\sigma^a\bar{\eta})\bar{\sigma}_a^{\dot{\alpha}\alpha}\bar{\mathrm{D}}_{\eta\dot{\alpha}} + d\eta^\alpha \\
&= -e^a \frac{i}{4}\bar{\sigma}_a^{\dot{\alpha}\alpha}\bar{\mathrm{D}}_{\eta\dot{\alpha}} + e^\alpha\,. \qquad (3.18)
\end{aligned}$$

IV. The Schwinger-Dyson Equation in the nonabelian case

Let us consider the supersymmetric Schwinger-Dyson equation for nonabelian gauge group $U(N_c)$. In the previous section both the one-form $\mathcal{A}$ and the chiral superfield $\mathcal{W}_\alpha$

were linear in the vector superfield V. In this case, it is natural to start with (3.6). In the nonabelian case, both $\mathcal{A}$ and $\mathcal{W}_\alpha$ are highly non-linear in the fundamental superfield V. The functional derivatives of V acting on the super Wilson-loop and the super Yang-Mills action become very complicated. It is technically difficult to obtain this way the nonabelian counterpart of eq. (3.17) in the last section which is composed only of the super Wilson-loops and the geometric operations. We must find another way to obtain the equation.

Let us pay attention to the left hand side of equation (B.5):

$$i\frac{1}{4g^2}\langle tr\{D_b v^{ba}(x)W[C_{xx}]\}\rangle \,. \tag{4.1}$$

This includes equation of motion for the Yang-Mills field v_a i.e. $D_b v^{ba}$. It is obtained by hitting the area derivative and subsequently the ordinary derivative on $\langle W_S[C]\rangle$. (See Appendix B).

In the previous section, the term $\mathrm{D}^\alpha\mathcal{W}_\alpha$ appeared in (3.10). As in (3.10), each component of $\mathrm{D}^\alpha\mathcal{W}_\alpha$ is composed of equations of motion for super abelian theory, and this is obtained by

$$\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^\alpha\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z)}\langle W_S[C]\rangle \,. \tag{4.2}$$

We infer that the final form of the supersymmetric equation should include

$$\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^\alpha\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z)}\langle trW_S[C]\rangle \tag{4.3}$$

on the lefthand side. We choose (4.3) our starting point for the nonabelian case. Using (2.17) and (2.11), we see that the above equation (4.3) becomes

$$\begin{aligned}
&\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^\alpha\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z)}\langle trW_S[C]\rangle \\
=\;& \frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^\alpha\langle tr\{\mathcal{F}_{a\dot{\alpha}}(z)W_S[C_{zz}]\}\rangle \\
=\;& \frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\langle tr(\mathcal{D}^\alpha\mathcal{F}_{a\dot{\alpha}}(z)W_S[C_{zz}])\rangle \\
=\;& -\frac{i}{4g^2}\langle tr\{\mathcal{D}\mathcal{W}(z)W_S[C_{zz}]\}\rangle \,,
\end{aligned} \tag{4.4}$$

$$\text{where}\quad \mathcal{D}\mathcal{W} \equiv \mathrm{D}^\alpha\mathcal{W}_\alpha - \{\mathcal{A}^\alpha,\mathcal{W}_\alpha\}\,, \tag{4.5}$$

$$\mathcal{F}_{a\dot{\alpha}} = \frac{i}{2}\sigma_{a\alpha\dot{\alpha}}\mathcal{W}^\alpha \,. \tag{4.6}$$

The expression $\mathcal{DW}$ is the nonabelian extension of $\mathrm{D}^\alpha \mathcal{W}_\alpha$. The components are equations of motion of the super Yang-Mills theory. Under the transformation of V, (C.25),

$$\mathrm{e}^{\mathrm{V}} \longrightarrow \mathrm{e}^{\mathrm{V}'} = \mathrm{e}^{\Lambda^\dagger} \mathrm{e}^{\mathrm{V}} \mathrm{e}^{\Lambda} \,, \tag{4.7}$$

the superfield (4.5) transforms covariantly,

$$\mathcal{DW} \longrightarrow (\mathcal{DW})' = \mathrm{e}^{-\Lambda} \mathcal{DW} \mathrm{e}^{\Lambda} \tag{4.8}$$

The expression (4.4) is locally super-gauge invariant. In order to make further manipulations manageable, we employ the Wess-Zumino gauge condition, which eliminates the fields $(\mathrm{C}, \chi, \bar{\chi}, \mathrm{M}, \mathrm{N})$. We will mainly use the coordinates $(y^a = x^a + i\theta\sigma^a\bar{\theta}, \theta, \bar{\theta})$ as the gauge covariance becomes clearer with these by the following reason. The chiral superfield Λ is the parameter of the restricted supergauge transformation. The imaginary part u of the lowest component of Λ corresponds to the usual local gauge parameter depending on the spacetime coordinates. In the Wess-Zumino gauge, only u of Λ survives as gauge parameters, in terms of $(y, \theta, \bar{\theta})$. The transformations (C.28) and (C.19) become

$$\begin{aligned} \mathcal{A}_A &\longrightarrow \mathcal{A}'_A = -\mathrm{e}^{iu(y)} \mathrm{D}_A \mathrm{e}^{-iu(y)} + \mathrm{e}^{iu(y)} \mathcal{A}_A \mathrm{e}^{-iu(y)} \,, \\ \mathcal{W}_\alpha &\longrightarrow \mathcal{W}'_\alpha = \mathrm{e}^{iu(y)} \mathcal{W}_\alpha \mathrm{e}^{-iu(y)} \,, \end{aligned} \tag{4.9}$$

We first calculate the one-form $\mathcal{A}$ and the chiral superfield $\mathcal{W}_\alpha$ in the Wess-Zumino gauge, where

$$\begin{aligned} \mathrm{V} &= -\theta\sigma^a\bar{\theta} v_a(x) + i\theta^2\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}^2\theta\lambda(x) \\ &\quad + \frac{1}{2}\theta^2\bar{\theta}^2 \mathrm{D}(x) \\ &= -\theta\sigma^a\bar{\theta} v_a(y) + i\theta^2\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}^2\theta\lambda(y) \\ &\quad + \frac{1}{2}\theta^2\bar{\theta}^2 \mathrm{D}(y) - \frac{i}{2}\theta^2\bar{\theta}^2 \partial_a v^a(y) \,, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \mathrm{e}^{\mathrm{V}} &= 1 + \mathrm{V} + \frac{1}{2}\mathrm{V}^2 \\ &= 1 - \theta\sigma^a\bar{\theta} v_a(y) + i\theta^2\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}^2\theta\lambda(y) \\ &\quad + \frac{1}{2}\theta^2\bar{\theta}^2 \mathrm{D}(y) - \frac{1}{4}\theta^2\bar{\theta}^2 v^2(y) - \frac{i}{2}\theta^2\bar{\theta}^2 \partial_a v^a(y) \,, \end{aligned} \tag{4.11}$$

$$\mathrm{e}^{-\mathrm{V}} = 1 - \mathrm{V} + \frac{1}{2}\mathrm{V}^2$$

$$\begin{aligned} &= 1+\theta\sigma^a\bar\theta v_a(y) - i\theta^2\bar\theta\bar\lambda(y) + i\bar\theta^2\theta\lambda(y) \\ &\quad -\frac{1}{2}\theta^2\bar\theta^2\mathrm{D}(y) + \frac{1}{4}\theta^2\bar\theta^2 v^2(y) - \frac{i}{2}\theta^2\bar\theta^2\partial_a v^a(y)\,. \end{aligned} \tag{4.12}$$

Using the expressions of D_α, $\bar{\mathrm{D}}_{\dot\alpha}$ in terms of $(y,\theta,\bar\theta)$, we obtain $\mathcal{A}_\alpha$ and $\mathcal{A}_a$:

$$\begin{aligned} \xi^\alpha\mathcal{A}_\alpha &= -\mathrm{e}^{-\mathrm{V}}\xi^\alpha\mathrm{D}_\alpha\mathrm{e}^{\mathrm{V}} \\ &= \xi\sigma^a\bar\theta v_a(y) - 2i\xi\theta\bar\theta\bar\lambda(y) + i\bar\theta^2\xi\lambda(y) \\ &\quad -\xi\theta\bar\theta^2\mathrm{D}(y) + i\bar\theta^2\xi\sigma^{ab}\theta v_{ab}(y) \\ &\quad -\theta^2\bar\theta^2\xi\sigma^a D_a\bar\lambda(y)\,, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \mathcal{A}_a &= \frac{i}{4}\bar\sigma_a^{\dot\alpha\alpha}\bar{\mathrm{D}}_{\dot\alpha}\mathrm{e}^{-\mathrm{V}}\mathrm{D}_\alpha\mathrm{e}^{\mathrm{V}} \\ &= -\frac{i}{2}v_a(y) - \frac{1}{2}\bar\lambda(y)\bar\sigma_a\theta + \frac{1}{2}\bar\theta\bar\sigma_a\lambda(y) \\ &\quad +\frac{i}{2}\bar\theta\bar\sigma_a\theta\mathrm{D}(y) + \frac{1}{2}\bar\theta\bar\sigma_a\sigma^{bc}\theta v_{bc}(y) \\ &\quad +\frac{i}{2}\theta^2\bar\theta\bar\sigma_a\sigma^b D_b\bar\lambda(y)\,, \end{aligned} \tag{4.14}$$

where we have defined $v_{ab}(y)$ and $D_a\lambda(y)$ respectively by

$$\begin{aligned} v_{ab}(y) &= \partial_a v_b(y) - \partial_b v_a(y) + \frac{i}{2}[v_a(y), v_b(y)]\,, \\ D_a\lambda(y) &= \partial_a\lambda(y) + \frac{i}{2}[v_a(y), \lambda(y)]\,. \end{aligned}$$

From the gauge transformation of $\mathcal{A}_a$ given by (4.9) and the explicit expression (4.14), we find

$$\begin{aligned} v_a(y) &\longrightarrow v'_a(y) = -2iU(y)\partial_a U^{-1}(y) + U(y)v_a(y)U^{-1}(y)\,, \\ \lambda(y) &\longrightarrow \lambda'(y) = U(y)\lambda(y)U^{-1}(y)\,, \\ \bar\lambda(y) &\longrightarrow \bar\lambda'(y) = U(y)\bar\lambda(y)U^{-1}(y)\,, \\ \mathrm{D}(y) &\longrightarrow \mathrm{D}'(y) = U(y)\mathrm{D}(y)U^{-1}(y)\,, \end{aligned}$$

where $U(y)$ denotes $\mathrm{e}^{iu(y)}$.

The chiral superfield $\mathcal{W}_\alpha$ takes the form

$$\xi^\alpha\mathcal{W}_\alpha = -\frac{1}{4}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{e}^{-\mathrm{V}}\xi^\alpha\mathrm{D}_\alpha\mathrm{e}^{\mathrm{V}}$$

$$
\begin{aligned}
&= -i\xi\lambda(y) - i\xi\sigma^{ab}\theta v_{ab}(y) + \xi\theta \mathrm{D}(y) \\
&\quad +\theta^2\xi\sigma^a D_a\bar{\lambda}(y)\,.
\end{aligned} \tag{4.15}
$$

Note that $\bar{\mathrm{D}}_{\dot{\alpha}}\mathcal{W}_\alpha = 0$ is trivially satisfied.

Inserting (4.13) into the one-form $\mathcal{A}$ given by (2.2), we obtain

$$
\begin{aligned}
\mathcal{A} &= dy^a\{-\frac{i}{2}v_a(y) - \frac{1}{2}\bar{\lambda}(y)\bar{\sigma}_a\eta + \frac{1}{2}\bar{\eta}\bar{\sigma}_a\lambda(y) \\
&\quad +\frac{i}{2}\bar{\eta}\sigma_a\eta\mathrm{D}(y) + \frac{1}{2}\bar{\eta}\bar{\sigma}_a\sigma^{bc}\eta v_{bc}(y) + \frac{i}{2}\eta^2\bar{\eta}\bar{\sigma}_a\sigma^b D_b\bar{\lambda}(y)\} \\
&\quad +\bar{\eta}^2\{-id\eta\lambda(y) + d\eta\eta\mathrm{D}(y) - id\eta\sigma^{ab}\eta v_{ab} \\
&\quad +\eta^2 d\eta\sigma^a D_a\bar{\lambda}(y)\}\,.
\end{aligned} \tag{4.16}
$$

The super Yang -Mills action is derived from (3.1),

$$
\begin{aligned}
\mathrm{S}_{SYM} &= \frac{1}{4g^2}\int d^4y\, tr\{-i\lambda(y)\sigma^a D_a\bar{\lambda}(y) + \frac{1}{2}\mathrm{D}^2(y) \\
&\quad -\frac{1}{4}v_{ab}(y)v^{ab}(y) - \frac{i}{8}\epsilon^{abcd}v_{ab}(y)v_{cd}(y)\}
\end{aligned} \tag{4.17}
$$

Hereafter, we denote a closed loop C on the superspace by

$$
z = z(t) = (y(t), \eta(t), \bar{\eta}(t)), \quad 0 \le t \le 1\,. \tag{4.18}
$$

A point on the loop C is denoted by $z' = (y', \theta, \bar{\theta})$. Let us take functional derivatives with respect to $v_a(y')$, $\lambda(y')$, $\bar{\lambda}(y')$ and $\mathrm{D}(y')$ acting the action S_{SYM} and obtain equations of motion for these fields:

$$
\begin{aligned}
T^r\frac{\delta}{\delta v_a^{(r)}(y')}\mathrm{S}_{SYM} &= \frac{1}{4g^2}\left(D_b v^{ba}(y') - \frac{1}{2}\bar{\sigma}^{a\dot{\alpha}\alpha}\{\bar{\lambda}_{\dot{\alpha}}(y'), \lambda_\alpha(y')\}\right)\,, \\
T^r\frac{\delta}{\delta\lambda^{(r)\alpha}(y')}\mathrm{S}_{SYM} &= -\frac{i}{4g^2}(\sigma^a D_a\bar{\lambda}(y'))_\alpha\,, \\
T^r\frac{\delta}{\delta\bar{\lambda}_{\dot{\alpha}}^{(r)}(y')}\mathrm{S}_{SYM} &= -\frac{i}{4g^2}(\bar{\sigma}^a D_a\lambda(y'))^{\dot{\alpha}}\,, \\
T^r\frac{\delta}{\delta\mathrm{D}^{(r)}(y')}\mathrm{S}_{SYM} &= \frac{1}{4g^2}\mathrm{D}(y')\,,
\end{aligned} \tag{4.19}
$$

where T^r denotes the generators of the gauge group $U(N_c)$ and the summation over r is implied. The expression$\mathcal{DW}$ in (4.3) is given in the Wess-Zumino gauge by

$$
\mathcal{DW} = \mathrm{D}^\alpha\mathcal{W}_\alpha - \{\mathcal{A}^\alpha, \mathcal{W}_\alpha\}
$$

$$
\begin{aligned}
&= -2\mathrm{D}(y) - 2\bar\theta\bar\sigma^a D_a\lambda(y) + 2\theta\sigma^a D_a\bar\lambda(y) \\
&\quad -2\theta\sigma_b\bar\theta[D_a v^{ab}(y) - \frac{1}{2}\bar\sigma^{b\dot\alpha\alpha}\{\bar\lambda_{\dot\alpha}(y), \lambda_\alpha(y)\}] \\
&\quad +2i\theta\sigma^a\bar\theta D_a\mathrm{D}(y) - 2i\theta^2\bar\theta\bar\sigma^b\sigma^a D_b D_a\bar\lambda(y) \\
&\quad -2i\theta^2[\mathrm{D}(y), \bar\theta\bar\lambda(y)]\,. \qquad (4.20)
\end{aligned}
$$

Using (4.19) and (4.20), we can rewrite the right hand side of equation (4.3) as

$$
\begin{aligned}
& -\frac{i}{4g^2}\langle tr\{\mathcal{D}\mathcal{W}(z')W_S[C_{z'z'}]\}\rangle \\
= & -\frac{i}{4g^2}\int[dv_a d\lambda^\alpha d\bar\lambda_{\dot\alpha} d\mathrm{D}]\mathrm{e}^{i\mathrm{S}_{SYM}} tr\{\mathcal{D}\mathcal{W}(z')W_S[C_{z'z'}]\} \\
= & \;2\int[dv_a d\lambda^\alpha d\bar\lambda_{\dot\alpha} d\mathrm{D}] tr\{T^r\, W_S[C_{z'z'}]\}\frac{\delta}{\delta\mathrm{V}^{(r)}_{\mathrm{mod}}(z')}\mathrm{e}^{i\mathrm{S}_{SYM}} \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta}{\delta\mathrm{V}^{(r)}_{\mathrm{mod}}(z')} \;\equiv\; & \frac{\delta}{\delta\mathrm{D}^{(r)}(y')} + i\bar\theta_{\dot\alpha}\frac{\delta}{\delta\bar\lambda^{(r)}_{\dot\alpha}(y')} - i\theta^\alpha\frac{\delta}{\delta\lambda^{(r)\alpha}(y')} \\
& +\theta\sigma_a\bar\theta\frac{\delta}{\delta v^{(r)}_a(y')} - i\theta\sigma^a\bar\theta D'^{(rs)}_a\frac{\delta}{\delta\mathrm{D}^{(s)}(y')} \\
& -\theta^2(\bar\theta\bar\sigma^a)^\alpha D'^{(rs)}_a\frac{\delta}{\delta\lambda^{(s)\alpha}(y')} - i\theta^2\bar\theta\bar\lambda^{(rs)}(y')\frac{\delta}{\delta\mathrm{D}^{(s)}(y')}\,, \qquad (4.22)
\end{aligned}
$$

and

$$
\begin{aligned}
D'^{(rs)}_a &\equiv \delta^{rs}\frac{\partial}{\partial y'^a} + \frac{i}{2}v^{(rs)}_a(y')\,, \\
v^{(rs)}_a(y') &\equiv if^{rts}v^{(t)}_a(y')\,, \\
\bar\lambda^{(rs)}(y') &\equiv if^{rts}\bar\lambda^{(t)}(y')\,. \qquad (4.23)
\end{aligned}
$$

Note that $\frac{\delta}{\delta\mathrm{V}^{(r)}_{\mathrm{mod}}}$ does not contain the derivatives with respect to the fields $(\mathrm{C}, \chi, \bar\chi, \mathrm{M}, \mathrm{N})$, while $\frac{\delta}{\delta\mathrm{V}}$ defined by equation (3.4) does. The covariant derivative $D^{(rs)}_a$ and fields $\bar\lambda^{(rs)}$ appear in the expression of $\frac{\delta}{\delta\mathrm{V}^{(r)}_{\mathrm{mod}}}$. This is because, even in the Wess-Zumino gauge, neither $\mathcal{A}$ nor $\mathcal{W}_\alpha$ is linear in V in the nonabelian case.

The infinitesimal change of the one-form $\mathcal{A}$ yields

$$
\delta W_S[C_{z_1z_2}] = \int_0^1 W_S[C_{z_1z(t)}]\delta\mathcal{A}(z(t))W_S[C_{z(t)z_2}]\,, \qquad (4.24)
$$

where $\int_0^1$ denotes the integration over t which parameterizes the curve $C_{z_1 z_2}$ and the line element $\frac{dz^M}{dt}$ is included in the expression $\delta\mathcal{A}(z(t))$.

Let us carry out the partial integration of the functional variables $(v_a, \lambda, \bar{\lambda}, \mathrm{D})$ in (4.21). The functional derivative $\frac{\delta}{\delta V_{\mathrm{mod}}}$ acts on the Wilson-loop $W_S[C_{z'z'}]$. Using (4.24), we find

$$\begin{aligned} & 2\int [dv_a d\lambda^\alpha d\bar{\lambda}_{\dot{\alpha}} d\mathrm{D}]\, tr\{T^r\, W_S[C_{z'z'}]\} \frac{\delta}{\delta \mathrm{V}^{(r)}_{\mathrm{mod}}(z')} \mathrm{e}^{i\mathrm{S}_{SYM}} \\ = \; & -2\int [dv_a d\lambda^\alpha d\bar{\lambda}_{\dot{\alpha}} d\mathrm{D}] \mathrm{e}^{i\mathrm{S}_{SYM}} \\ & \qquad \times \oint tr\{T^r\, W_S[C_{z'z}] \frac{\delta\mathcal{A}(z)}{\delta \mathrm{V}^{(r)}_{\mathrm{mod}}(z')} W_S[C_{zz'}]\}\,, \end{aligned} \tag{4.25}$$

where $\oint$ denotes the integration over z along the closed loop $C_{z'z'} = C$ and the line element dz^M is included in $\mathcal{A}(z)$ as in (4.24).

After some calculation from (4.16) and (4.22), we obtain

$$\frac{\delta}{\delta \mathrm{V}^{(r)}_{\mathrm{mod}}(z')} \mathcal{A}(z) = \mathcal{K}_y + \mathcal{K}_\eta\,, \tag{4.26}$$

$$\begin{aligned} \mathcal{K}_y \;\equiv\; & T^s\{-\frac{i}{2}\delta^{rs}(\eta-\theta)\sigma^a(\bar{\eta}-\bar{\theta}) \\ & -\frac{1}{2}dy^a\bar{\theta}\bar{\sigma}^b\sigma_a\bar{\eta}\eta^2\tilde{D}^{(sr)}_b - dy^b\bar{\eta}\bar{\sigma}_b\sigma^{ca}\eta\bar{\theta}\bar{\sigma}_a\theta\tilde{D}^{(sr)}_c \\ & -\frac{1}{2}dy^b\eta\sigma_b\bar{\eta}\bar{\theta}\bar{\sigma}^a\theta\tilde{D}^{(sr)}_a - \frac{1}{2}dy^b\theta^2\bar{\theta}\bar{\sigma}^a\sigma_b\bar{\eta}\tilde{D}^{(sr)}_a\}\delta^4(y-y')\,, \end{aligned} \tag{4.27}$$

$$\begin{aligned} \mathcal{K}_\eta \;\equiv\; & T^s\bar{\eta}^2\{\delta^{sr}(\eta d\eta - \theta d\eta) \\ & +i\eta^2 d\eta\sigma^a\bar{\theta}\tilde{D}^{(sr)}_a - 2i\theta\sigma_a\bar{\theta}d\eta\sigma^{ba}\eta\tilde{D}^{(sr)}_b \\ & +id\eta\eta\theta\sigma^a\bar{\theta}\tilde{D}^{(sr)}_a + i\theta^2 d\eta\sigma^a\bar{\theta}\tilde{D}^{(sr)}_a\}\delta^4(y-y') \end{aligned} \tag{4.28}$$

$$\tilde{v}_a \equiv v_a - i\bar{\lambda}\bar{\sigma}_a\eta\,, \tag{4.29}$$

$$\tilde{D}^{(sr)}_a \equiv \delta^{sr}\frac{\partial}{\partial y^a} + \frac{i}{2}\tilde{v}^{(sr)}_a(y) \tag{4.30}$$

where the summation over s is taken implicitly. Note that all derivatives in the above equations are with respect to y^a; they act on the delta function $\delta^4(y-y')$.

The right hand side of (4.25) as it is contains the terms depending on the fields $\tilde{v}_a = v_a - i\bar{\lambda}\bar{\sigma}_a\eta$ explicitly and is not represented by supersymmetric Wilson-loops alone. Our remaining task in this section is to show that these fields are generated from geometrical

operations acting on the super Wilson-loop Using (2.11), we find

$$
\begin{aligned}
& \frac{\partial}{\partial y^a} T^r W_S[C_{z'z}] T^r \delta^4(y-y') W_S[C_{zz'}] \\
= \ & T^r W_S[C_{z'z}] \{ T^r (\partial_a \delta^4(y-y')) W_S[C_{zz'}] - [\mathcal{A}_a(z), T^r] \delta^4(y-y') \} \\
= \ & T^r W_S[C_{z'z}] T^s \{ \mathcal{D}_a^{(sr)} \delta(y-y') \} W_S[C_{zz'}] \,, \qquad (4.31) \\
& \mathcal{D}_a^{(sr)} \equiv \delta^{sr} \frac{\partial}{\partial y^a} - \mathcal{A}_a^{(sr)}(z) \,. \qquad (4.32)
\end{aligned}
$$

Note that, in (4.26), $\frac{\delta}{\delta \mathrm{V}^{(r)}_{\mathrm{mod}}(z')} \mathcal{A}(z)$ depends on the fields v_a and $\bar{\lambda}_{\dot{\alpha}}$ only through their combination $\tilde{v}_a = v_a - i\bar{\lambda}\bar{\sigma}_a \eta$. From (4.14), we can express $\mathcal{A}_a(z)$ via $\tilde{v}_a$:

$$
\mathcal{A}_a(z) = -\frac{i}{2} \tilde{v}_a(y) - \bar{\eta}^{\dot{\alpha}} \bar{\mathrm{D}}_{\dot{\alpha}} \mathcal{A}_a(z) \,. \qquad (4.33)
$$

Furthermore, $\mathcal{A}_a(z)$ can be rewritten as

$$
\begin{aligned}
\mathcal{A}_a(z) &= -\frac{i}{2} \tilde{v}_a(y) - \frac{i}{4} \bar{\eta}^{\dot{\alpha}} \bar{\mathrm{D}}_{\dot{\alpha}} \bar{\sigma}_a^{\dot{\beta}\beta} \bar{\mathrm{D}}_{\dot{\beta}} \mathrm{e}^{-\mathrm{V}} \mathrm{D}_\beta \mathrm{e}^{\mathrm{V}} \\
&= -\frac{i}{2} \tilde{v}_a(y) + \frac{i}{8} \bar{\eta}^{\dot{\alpha}} \bar{\sigma}_a^{\dot{\beta}\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\mathrm{D}} \bar{\mathrm{D}} \mathrm{e}^{-\mathrm{V}} \mathrm{D}_\beta \mathrm{e}^{\mathrm{V}} \\
&= -\frac{i}{2} \tilde{v}_a(y) + \frac{1}{2} \bar{\eta} \bar{\sigma}_a \mathcal{W}(z) \,. \qquad (4.34)
\end{aligned}
$$

Here, in the last equality, we have used (C.26). Using this expression, we can write (4.26) in terms of $\mathcal{A}_a(z)$ and $\mathcal{W}_\alpha(z)$ without employing $\tilde{v}_a$,

$$
\begin{aligned}
& \frac{\delta}{\delta \mathrm{V}^{(r)}_{\mathrm{mod}}(z')} \mathcal{A}(z) \\
= \ & T^s \{ -\frac{i}{2} \delta^{rs} dy^a (\eta - \theta) \sigma_a (\bar{\eta} - \bar{\theta}) - \frac{1}{2} dy^a \delta(\eta - \theta) \bar{\theta} \bar{\sigma}^b \sigma_a \bar{\eta} \mathcal{D}_b^{(sr)} \\
& + \frac{i}{4} \bar{\eta}^2 dy^a \bar{\theta} \bar{\sigma}_a \mathcal{W}^{(sr)} \\
& + \bar{\eta}^2 \left(d\eta (\eta - \theta) \delta^{rs} + i d\eta \sigma^a \bar{\theta} \delta(\eta - \theta) \mathcal{D}_a^{(sr)} \right) \} \delta^4(y - y') \,. \qquad (4.35)
\end{aligned}
$$

We see that all field dependent terms are expressed through $\mathcal{D}_a^{(sr)}$ and $\mathcal{W}_\alpha$. Note that, in equation (4.34), $\mathcal{W}_\alpha$ is multiplied by $\bar{\eta}$, while $\mathcal{K}_\eta$ defined by (4.28) is multiplied by $\bar{\eta}^2$. So $\mathcal{W}_\alpha$ originates from $\mathcal{K}_y$ only.

Remember the relation between $\mathcal{W}_\alpha$ and $\mathcal{F}_{a\dot{\alpha}}$ given by (4.6) and that $\mathcal{F}_{a\dot{\alpha}}$ is obtained from area derivatives by acting on the super Wilson-loop. The right hand side of (4.25) can, therefore, be expressed with the derivative ∂_a and the area derivative of the super

Wilson-loop alone. We conclude

$$
\begin{aligned}
&\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^{\alpha}\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}\langle trW_S[C]\rangle \\
= \quad & -2\oint\{-\frac{i}{2}dy^a(\eta-\theta)\sigma^a(\bar{\eta}-\bar{\theta})-\frac{1}{2}dy^a\delta(\eta-\theta)\bar{\theta}\bar{\sigma}^b\sigma_a\bar{\eta}\partial_b \\
& \quad +\bar{\eta}^2\Big(d\eta(\eta-\theta)+id\eta\sigma^a\bar{\theta}\delta(\eta-\theta)\partial_a\Big)\}\langle tr\,W_S[C_{z'z}]\delta^4(y-y')tr\,W_S[C_{zz'}]\rangle \\
& +2\oint\frac{1}{8}\bar{\eta}^2\delta(\eta-\theta)\delta^4(y-y')dy^a(\bar{\sigma}^b\sigma_a\bar{\theta})^{\dot{\alpha}} \\
& \times\Big\{\langle(\frac{\delta}{\delta\Sigma^{b\dot{\alpha}}(z)}tr\,W_S[C_{z'z}])tr\,W_S[C_{zz'}]\rangle-\langle tr\,W_S[C_{z'z}](\frac{\delta}{\delta\Sigma^{b\dot{\alpha}}(z)}tr\,W_S[C_{zz'}])\rangle\Big\}
\end{aligned}
\tag{4.36}
$$

Here we have used the completeness relation obeyed by the generators T^r of $U(N_c)$.

We have obtained the supersymmetric Schwinger-Dyson equation for the nonabelian gauge groups (4.36) in a closed form with respect to the super Wilson-loops. Eq.(4.36) is invariant under the ordinary gauge transformation given by (4.9). The invariance of the left hand side is obvious while that of the right hand side is less obvious. It contains the delta function $\delta^4(y-y')$. At the nonvanishing support, the right hand side is invariant as well.

V. Manifestly Supersymmetric Form of the Schwinger-Dyson Equation

One may wonder whether (4.36) is consistent with (3.17) at $N_c = 1$, namely, when we take the abelian reduction of the nonabelian equation. We have

$$
\begin{aligned}
&\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^{\alpha}\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}\langle W_S[C]\rangle \\
= \quad & -2\langle W_S[C]\rangle\oint\{-\frac{i}{2}dy^a(\eta-\theta)\sigma_a(\bar{\eta}-\bar{\theta}) \\
& \quad -\frac{1}{2}dy^a\delta(\eta-\theta)\bar{\theta}\bar{\sigma}^b\sigma_a\bar{\eta}\partial_b \\
& \quad +\bar{\eta}^2(d\eta(\eta-\theta)+id\eta\sigma^a\bar{\theta}\delta(\eta-\theta)\partial_a)\}\delta^4(y-y')\,.
\end{aligned}
\tag{5.1}
$$

After some calculations, we see that the right hand side of (5.1) is not algebraically equal to that of (3.17). Are these two equations still consistent with each other ? We will be able to answer to this question in the affirmative.

Let us consider

$$\mathcal{I}_{\rm R} = \mathcal{D}_f^\alpha {\rm D}_\alpha \delta(\eta-\theta)\delta(\bar\eta-\bar\theta) f(z) \tag{5.2}$$

where

$$\begin{aligned} f(z) &\equiv \langle T^r W_S[C_{z'z}] T^r \delta^4(y-y') W_S[C_{zz'}]\rangle \,, \\ {\rm D}_\alpha &= \frac{\partial}{\partial \eta^\alpha} - 2i(\sigma^a\bar\eta)_\alpha \frac{\partial}{\partial y^a} \,, \\ \text{and} \quad \bar{\rm D}_{\dot\alpha} &= -\frac{\partial}{\partial \bar\eta^{\dot\alpha}} \,, \end{aligned}$$

and $\mathcal{D}_f^\alpha$ are given by (3.18). (Recall that we use the same notation for z and z', $z = (y,\eta,\bar\eta)$, $z' = (y',\theta,\bar\theta)$ as before.) After some calculation, we find that $\mathcal{I}_{\rm R}$ can be decomposed into

$$\mathcal{I}_{\rm R} = \mathcal{I}_{\rm W-Z} + \mathcal{I}_{\rm d} \,, \tag{5.3}$$

$$\begin{aligned} \mathcal{I}_{\rm W-Z} = \; & \{ i(\eta-\theta)\sigma_a(\bar\eta-\bar\theta) dy^a + \bar\theta\bar\sigma^b\sigma_a\bar\eta\delta(\eta-\theta) dy^a \partial_b \\ & -2\bar\eta^2 \{ d\eta(\eta-\theta) + i d\eta\sigma^a\bar\theta\delta(\eta-\theta)\partial_a \} f(z) \\ & -\frac{i}{2}\bar\eta^2\delta(\eta-\theta) dy^a \langle T^r W_S[C_{z'z}]\delta^4(y-y')[\theta\bar\sigma_a\mathcal{W}, T^r] W_S[C_{zz'}]\rangle \,, \end{aligned} \tag{5.4}$$

$$\begin{aligned} \mathcal{I}_{\rm d} = \; & \bar\theta^2 \{ 2 d\eta(\eta-\theta) + \delta(\eta-\theta) dy^a \partial_a \} f(z) \\ & -\bar\theta^2\bar\eta^2\delta(\eta-\theta)\langle T^r W_S[C_{z'z}]\delta^4(y-y')[d\eta\mathcal{W}, T^r] W_S[C_{zz'}]\rangle \,. \end{aligned} \tag{5.5}$$

In these calculations, $\mathcal{W}_\alpha$ comes from two origins. Let us consider the derivative $\bar{\rm D}_{\dot\alpha}\partial_a$ with this order and act on $f(z)$. First ∂_a acts on $f(z)$, which generates $\mathcal{A}_a$ through W_S in $f(z)$. The subsequent action of $\bar{\rm D}_{\dot\alpha}$ generates $\bar{\rm D}_{\dot\alpha}\mathcal{A}_a = -\mathcal{F}_{a\dot\alpha}$ and $\mathcal{W}_\alpha$. (Note that $\partial_a \bar{\rm D}_{\dot\alpha} f(z) = 0$ because $\bar{\rm D}_{\dot\alpha} W_S = 0$.) The $\mathcal{W}_\alpha$ also appears when the derivative $\frac{\partial}{\partial\eta^\alpha}$ acts on $f(z)$. This action yields $\mathcal{A}_{\underline{\alpha}}$ which has the Einstein index. From the definition (2.2), $\mathcal{A}_{\underline{\alpha}}$ can be expressed with $\mathcal{A}_a$ and $\mathcal{A}_\alpha$ by

$$\mathcal{A}_{\underline{\alpha}} = \mathcal{A}_\alpha - 2i(\sigma^a\bar\eta)_\alpha \mathcal{A}_a \,.$$

Using the explicit expressions for $\mathcal{A}_\alpha$ and $\mathcal{A}_a$ in the Wess-Zumino gauge, we find

$$\mathcal{A}_{\underline{\alpha}} = \bar\eta^2 \{ -i\lambda_\alpha(y) + \eta_\alpha {\rm D}(y) - i(\sigma^{ab}\eta)_\alpha v_{ab} + \eta^2(\sigma^a D_a\bar\lambda)_\alpha \} \,.$$

Comparing this with (4.15), we find the following relations between $\mathcal{A}_{\underline{\alpha}}$ and $\mathcal{W}_\alpha$:

$$\mathcal{A}_{\underline{\alpha}} = \bar\eta^2 \mathcal{W}_\alpha \,. \tag{5.6}$$

The $\mathcal{W}_\alpha$ appears when $\frac{\partial}{\partial\eta^\alpha}$ acts on $f(z)$.

Returning to (5.3), we consider taking the trace of the first term $\mathcal{I}_{\mathrm{W-Z}}$ and integrate this along the closed loop C, $I_{\mathrm{W-Z}} \equiv \oint tr\mathcal{I}_{\mathrm{W-Z}}$. Since $\mathcal{W}_\alpha$ is obtained by the area derivative on the Wilson-loop $W_S[C_{z'z}]$ or $W_S[C_{zz'}]$, $I_{\mathrm{W-Z}}$ can be written with the super Wilson-loop and its area derivative alone as

$$\begin{aligned} I_{\mathrm{W-Z}} &= \oint\{i(\eta-\theta)\sigma_a(\bar\eta-\bar\theta)dy^a + \bar\theta\bar\sigma^b\sigma_a\bar\eta\delta(\eta-\theta)dy^a\partial_b \\ &\quad -2\bar\eta^2 d\eta(\eta-\theta) - 2i\bar\eta^2 d\eta\sigma^a\bar\theta\partial_a\}\langle tr\, W_S[C_{z'z}]\delta^4(y-y')tr\, W_S[C_{zz'}]\rangle \\ &\quad +2\oint\frac{1}{8}\bar\eta^2\delta(\eta-\theta)\delta^4(y-y')dy^a(\bar\sigma^b\sigma_a)^{\dot\alpha} \\ &\quad \times\left\{\langle(\frac{\delta}{\delta\Sigma^{b\dot\alpha}(z)}tr\, W_S[C_{z'z}])tr\, W_S[C_{zz'}]\rangle - \langle tr\, W_S[C_{z'z}](\frac{\delta}{\delta\Sigma^{b\dot\alpha}(z)}tr\, W_S[C_{zz'}])\rangle\right\} \end{aligned}$$

This $I_{\mathrm{W-Z}}$ is exactly the right hand side of the supersymmetric Schwinger-Dyson equation which we have obtained in the previous section. We find

$$\begin{aligned} &\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar\sigma^{a\dot\alpha\beta}\mathrm{D}^\alpha\frac{\delta}{\delta\Sigma^{a\dot\alpha}(z')}\langle tr\, W_S[C]\rangle \\ &= \oint tr\,\mathcal{I}_{\mathrm{R}} - \oint tr\,\mathcal{I}_{\mathrm{d}} \\ &= \oint \mathcal{D}^\alpha_f \mathrm{D}_\alpha\delta(z-z')\langle tr\, T^r W_S[C_{z'z}]T^r W_S[C_{zz'}]\rangle - I_{\mathrm{d}}\,, \\ &I_{\mathrm{d}} \equiv \oint tr\,\mathcal{I}_{\mathrm{d}}\,. \end{aligned} \tag{5.7}$$

In the abelian reduction of the above equation, the first term of the right hand side is the same as that of eq. (3.17).

This fact indicates that I_{d} must identically be zero if the nonabelian equation is consistent with the abelian case. This is the case in fact. From the relation $\mathcal{A}_{\underline{\alpha}} = \bar\eta^2\mathcal{W}_\alpha$ and (2.10),

$$\begin{aligned} I_{\mathrm{d}} &= \frac{1}{2}\bar\theta^2\oint\{dy^a\frac{\partial}{\partial y^a} + d\eta^\alpha\frac{\partial}{\partial\eta^\alpha}\}\delta(\eta-\theta)\delta^4(y-y') \\ &\qquad \times\langle tr\, T^r W_S[C_{z'z}]T^r W_S[C_{zz'}]\rangle \\ &= \frac{1}{2}\bar\theta^2\oint\{dy^a\frac{\partial}{\partial y^a} + d\eta^\alpha\frac{\partial}{\partial\eta^\alpha} + d\bar\eta^{\dot\alpha}\frac{\partial}{\partial\bar\eta^{\dot\alpha}}\}\delta(\eta-\theta)\delta^4(y-y') \\ &\qquad \times\langle tr\, T^r W_S[C_{z'z}]T^r W_S[C_{zz'}]\rangle \\ &= \frac{1}{2}\bar\theta^2\oint dz^M\frac{\partial}{\partial z^M}\delta(\eta-\theta)\delta^4(y-y')\langle tr\, T^r W_S[C_{z'z}]T^r W_S[C_{zz'}]\rangle\,, \end{aligned} \tag{5.8}$$

where we have used $\bar{\mathrm{D}}_{\dot\alpha} W_S = -\frac{\partial}{\partial\bar\eta^{\dot\alpha}} W_S = 0$ in the second equality. This equation is the contour integral along the closed loop C in superspace of the total derivative and must vanish.

This identity is related to the restricted supergauge transformations which we have explained in Appendix C. Consider

$$\mathcal{Z}[C_{z'z'}] = \langle W_S[C_{z'z'}]\rangle = \int [d\mathrm{V}] \mathrm{e}^{i\mathrm{S}_{SYM}} W_S[C_{z'z'}]\,. \tag{5.9}$$

Here we do not take the trace of $W_S[C_{z'z'}]$ and the integrand should be understood in the sense of matrix elements. Under the restricted supergauge transformations, the one-form $\mathcal{A}$ transforms as

$$\delta_\Lambda \mathcal{A}_M = -\partial_M \Lambda + [\mathcal{A}_M, \Lambda]\,. \tag{5.10}$$

Using (4.24), we obtain

$$\begin{aligned} 0 &= \oint \langle W_S[C_{z'z}]\delta_\Lambda \mathcal{A}(z) W_S[C_{zz'}]\rangle \\ &= -\oint dz^M \frac{\partial}{\partial z^M} \langle W_S[C_{z'z}]\Lambda(z) W_S[C_{zz'}]\rangle\,. \end{aligned} \tag{5.11}$$

In terms of $(y, \eta, \bar\eta)$, $\frac{\partial}{\partial z^{M=\dot\alpha}} = \frac{\partial}{\partial\bar\eta^{\dot\alpha}} = -\bar{\mathrm{D}}_{\dot\alpha}$ and $\bar{\mathrm{D}}_{\dot\alpha} W_S = 0$ as well as $\bar{\mathrm{D}}_{\dot\alpha}\Lambda = 0$. We find that the above identity becomes

$$0 = \oint \{dy^a \frac{\partial}{\partial y^a} + d\eta^\alpha \frac{\partial}{\partial \eta^\alpha}\} \langle W_S[C_{z'z}]\Lambda(z) W_S[C_{zz'}]\rangle\,. \tag{5.12}$$

Acting on eq. (5.12) the functional derivative $\frac{\delta}{\delta\Lambda^{(r)}(z')}$, which satisfies

$$\frac{\delta}{\delta\Lambda^{(r)}(z')}\Lambda(z) = \frac{1}{4}\bar{\mathrm{D}}'^2 \delta(z - z') T^r = \frac{1}{4}\bar{\mathrm{D}}^2 \delta(z - z') T^r\,,$$

we find

$$\begin{aligned} 0 &= \frac{1}{4}\oint \{dy^a \frac{\partial}{\partial y^a} + d\eta^\alpha \frac{\partial}{\partial \eta^\alpha}\} \langle W_S[C_{z'z}]\{\bar{\mathrm{D}}'^2 \delta(z - z')\} T^r W_S[C_{zz'}]\rangle \\ &= \oint \{dy^a \frac{\partial}{\partial y^a} + d\eta^\alpha \frac{\partial}{\partial \eta^\alpha}\} \langle W_S[C_{z'z}]\delta(\eta - \theta)\delta^4(y - y') T^r W_S[C_{zz'}]\rangle\,. \end{aligned}$$

Multiplying this equation by the generator T^r and taking its trace, we see that the right hand side is equal to I_{d}.

Inserting $I_{\mathrm{d}} = 0$ into (5.7), multiplying the both sides by the gauge volume which we have thrown away in taking the Wess-Zumino gauge condition, we find the final form of the

nonabelian supersymmetric Schwinger-Dyson equation for the super Wilson-loop:

$$\begin{aligned}
&\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^{\alpha}\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}\langle tr\,W_S[C]\rangle \\
= \quad & \oint \mathcal{D}_f^{\alpha}\mathrm{D}_{\alpha}\delta(z-z')\langle tr\,W_S[C_{z'z}]tr\,W_S[C_{zz'}]\rangle\;, \\
&\mathcal{D}_f^{\alpha} = -e^{a}\frac{i}{4}\bar{\sigma}_a^{\dot{\alpha}\alpha}\bar{\mathrm{D}}_{\eta\dot{\alpha}} + e^{\alpha}\;, \\
&\langle\cdots\rangle = \int[d\mathrm{V}]\,\mathrm{e}^{i\mathrm{S}_{SYM}}\,\cdots\;, \qquad (5.13)
\end{aligned}$$

where we have used the closure property (B.11) of $U(N_c)$. This expression is manifestly invariant under the restricted supergauge transformations. The final equation obtained is the first among the infinite number of the Schwinger-Dyson equations which exhaust the dynamics of supersymmetric Yang-Mills theory. At the same time, it exhibits string dynamics suggested by the theory. In fact, the right hand side tells us that the loop C splits into two parts by the dynamics if two points on the loop coalesce at z'; this latter condition comes from the delta function.

Let us now derive the rest of the Schwinger-Dyson equations. This time, not only splitting of a loop but also joining of two loops can take place from the original configuration $<\prod_{i=1}^{n} trW_S[C^{(i)}]>$. The functional derivative $\frac{\delta}{\delta\mathrm{V}_{\mathrm{mod}}^{(r)}(z')}\mathcal{A}(z)\quad z'\in C^{(j)}$, which appeared in the derivation of our equation in the previous section, now acts on every loop including the original one $C^{(j)}$. We find

$$\begin{aligned}
&\frac{1}{8g^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^{\alpha}\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}\langle\prod_{i=1}^{n}trW_S[C^{(i)}]\rangle \\
&=\sum_{j=1}^{n}\theta(z'\in C^{(j)})\langle\left(\prod_{i\neq j}^{n}trW_S[C^{(i)}]\right)\oint_{C^{(j)}}\mathcal{D}_f^{(j)\alpha}\mathrm{D}_{\alpha}^{(j)}\delta(z_j-z')trW_S[C^{(i)}_{z'z_j}]trW_S[C^{(i)}_{z_jz'}]\,\rangle \\
&+\sum_{j=1}^{n}\theta(z'\in C^{(j)})\sum_{k\neq j}^{n}\langle\left(\prod_{i\neq j,k}^{n}trW_S[C^{(i)}]\right)\oint_{C^{(k)}}\mathcal{D}_f^{(k)\alpha}\mathrm{D}_{\alpha}^{(k)}\delta(z_k-z')trW_S[C^{(k)}_{z_kz_k}+C^{(j)}_{z'z'}]\,\rangle\,.
\end{aligned}$$

(5.14)

Here, the step function $\theta(z'\in C^{(j)})$ indicates the case that z' is on $C^{(j)}$. As before, the second line of this equation describes the splitting of an individual loop into two if two points on the loop coalesce at z'. The third line describes the joining of two loops into one in the case where the point z' is shared by the two loops. This latter condition is again a consequence of the delta function.

As for the large N_c limit with $g_c^2 \equiv N_c g^2$ kept finite, we obtain the supersymmetric

extension of the Migdal-Makeenko equation which is summarized in Appendix B:

$$\begin{aligned}
&\frac{1}{8g_c^2}\epsilon_{\alpha\beta}\bar{\sigma}^{a\dot{\alpha}\beta}\mathrm{D}^{\alpha}\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}\mathbf{W}_S[C] \\
= \;& \oint \mathcal{D}_f^{\alpha}\mathrm{D}_{\alpha}\{\delta(z-z')\mathbf{W}_S[C_{z'z}]\mathbf{W}_S[C_{zz'}]\}\,, \\
&\mathbf{W}_S[C_{z_1z_2}] \equiv \langle\frac{1}{N_c}tr\,W_S[C_{z_1z_2}]\rangle = \int[d\mathrm{V}]\,\mathrm{e}^{i\mathrm{S}_{SYM}}tr\,W_S[C_{z_1z_2}]\,. \qquad (5.15)
\end{aligned}$$

The dynamics is then formally contained in the one-point average[3].

Eqs. (5.13) and (5.15) are both manifestly supersymmetric. The supersymmetry transformations are understood as coordinate transformations in superspace:

$$z = (y^a, \eta, \bar{\eta}) \longrightarrow z' = (y^a + 2i\eta\sigma^a\bar{\xi}, \eta+\xi, \bar{\eta}+\bar{\xi})\,, \qquad (5.16)$$

where ξ and $\bar{\xi}$ denote the infinitesimal Grassmann parameters for the transformation. The derivatives D_A having flat indices are invariant under the coordinate transformations; they commute or anticommute with the differential operator Q, $\bar{\mathrm{Q}}$. The bases e^A are invariant under the coordinate transformations as well,

$$\begin{aligned}
&e^a = dy^a - 2id\eta\sigma^a\bar{\eta} \\
\longrightarrow\;& e'^a = dy' - 2id\eta'\sigma^a\bar{\eta}' \\
&\qquad = d(y^a + 2i\eta\sigma^a\bar{\xi}) - 2id\eta\sigma^a(\bar{\eta}+\bar{\xi}) \\
&\qquad = e^a\,, \\
&e^{\alpha} \longrightarrow e^{\alpha}\,,\quad e^{\dot{\alpha}} \longrightarrow e^{\dot{\alpha}}\,. \qquad (5.17)
\end{aligned}$$

The delta function $\delta(z-z') = \delta(\eta-\theta)\delta(\bar{\eta}-\bar{\theta})\delta^4(y-y')$ is obviously invariant.

We also obtain

$$\frac{\delta}{\delta\Sigma^{\alpha\beta}(z')}\langle tr\,W_S[C]\rangle = 0\,,\; \frac{\delta}{\delta\Sigma^{\alpha\dot{\alpha}}(z')}\langle tr\,W_S[C]\rangle = 0\,,\; \frac{\delta}{\delta\Sigma^{\dot{\alpha}\dot{\beta}}(z')}\langle tr\,W_S[C]\rangle = 0\,. \qquad (5.18)$$

from eq. (2.18). These equations are identities in the original variables but should be treated as constraints as soon as we employ the super Wilson-loop as a fundamental variable.

In the large N_c limit (5.18) becomes

$$\frac{\delta}{\delta\Sigma^{\alpha\beta}(z')}\mathbf{W}_S[C] = 0\,,\; \frac{\delta}{\delta\Sigma^{\alpha\dot{\alpha}}(z')}\mathbf{W}_S[C] = 0\,,\; \frac{\delta}{\delta\Sigma^{\dot{\alpha}\dot{\beta}}(z')}\mathbf{W}_S[C] = 0\,. \qquad (5.19)$$

[3] There may be a subtlety in taking the large N_c limit in the case of supersymmetric gauge theories. This is related to the existence of degenerate vacua and the validity of the cluster property.

VI. Solution in the abelian case

In this section, we solve the supersymmetric abelian Schwinger-Dyson equation. Our main objective here is to check that our nonlinear functional equation does contain the desirable nontrivial solutions at least in the linearized approximation. Start with

$$\frac{1}{8g^2}\epsilon_{\alpha\beta}\mathrm{D}^\alpha\bar{\sigma}^{a\dot{\alpha}\alpha}\frac{\delta}{\delta\Sigma^{a\dot{\alpha}}(z')}\ln\mathbf{W}_S[C]=\oint\mathcal{D}^\alpha_f\mathrm{D}_\alpha\delta(z-z')\,, \tag{6.1}$$

$$\frac{\delta}{\delta\Sigma^{\alpha\beta}(z')}\ln\mathbf{W}_S[C]=0\,,$$

$$\frac{\delta}{\delta\Sigma^{\dot{\alpha}\dot{\beta}}(z')}\ln\mathbf{W}_S[C]=0\,,$$

$$\frac{\delta}{\delta\Sigma^{\alpha\dot{\alpha}}(z')}\ln\mathbf{W}_S[C]=0\,, \tag{6.2}$$

which are respectively (3.17) and (5.19) divided for each side by $\mathbf{W}_S[C]$. These are the first order approximation to the perturbative expansion of the supersymmetric nonabelian Schwinger-Dyson equation.

We now take an ansatz for the $\ln\mathbf{W}_S[C]$:

$$\ln\mathbf{W}_S[C]=\oint e^A\oint e'^B\mathcal{P}_{BA}(z,z')\,,$$

$$e^A=e^A(s)\,,\ \ z=z(s)\,,\ e'^B=e^B(t)\,,\ \ z'=z(t)\,, \tag{6.3}$$

$$\left(\ln\mathbf{W}_S[C_{z_fz_i}]=\int_{z_i}^{z_f}e^A\int_{z_i}^{z_f}e'^B\mathcal{P}_{BA}(z,z')\right) \tag{6.4}$$

where $\oint$ denotes the contour integration along the closed loop C. We use s and t to parameterize the loop C. Exchanging s and t, we find the relation

$$\mathcal{P}_{BA}(z,z')=(-)^{|A||B|}\mathcal{P}_{AB}(z',z)\,. \tag{6.5}$$

Since the system given by (6.1) and (6.2) takes a manifestly supersymmetric form and the bases e^A are invariant under the supertransformations, $\mathcal{P}_{BA}(z,z')$ must take the following form:

$$\mathcal{P}_{BA}(z,z')=\mathcal{P}_{BA}(y^a-y'^a-i(\eta-\eta')\sigma^a(\bar{\eta}+\bar{\eta}'),\eta-\eta',\bar{\eta}-\bar{\eta}')=\mathcal{P}_{BA}(u^a,\tau,\bar{\tau})\,,$$

where we have introduced the new coordinates u^a, τ and $\bar{\tau}$ respectively by

$$u^a\equiv y^a-y'^a-i(\eta-\eta')\sigma^a(\bar{\eta}+\bar{\eta}')\,,\ \tau\equiv\eta-\eta'\,,\ \bar{\tau}\equiv\bar{\eta}-\bar{\eta}'\,. \tag{6.6}$$

These new coordinates are invariant under the supersymmetry transformations (5.16). Useful formulas for $\mathrm{D}_A(y,\eta,\bar{\theta})$ or $\mathrm{D}'_A(y',\eta',\bar{\eta}')$ in terms of u^a, τ, and $\bar{\tau}$ are

$$
\mathrm{D}_A \;=\; \begin{cases} \frac{\partial}{\partial y^a} = \frac{\partial}{\partial u^a} \ , \\ \mathrm{D}_\alpha = \frac{\partial}{\partial \eta^\alpha} + 2i(\sigma^a\bar{\eta})_\alpha \frac{\partial}{\partial y^a} = \frac{\partial}{\partial \tau^\alpha} + i(\sigma^a\bar{\tau})_\alpha \frac{\partial}{\partial u^a} \ , \\ \bar{\mathrm{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\eta}^{\dot{\alpha}}} = -\frac{\partial}{\partial \bar{\tau}^{\dot{\alpha}}} - i(\tau\sigma^a)_{\dot{\alpha}} \frac{\partial}{\partial u^a} \ . \end{cases}
$$

$$
\mathrm{D}'_A \;=\; \begin{cases} \frac{\partial}{\partial y'^a} = -\frac{\partial}{\partial u^a} \ , \\ \mathrm{D}'_\alpha = \frac{\partial}{\partial \eta'^\alpha} + 2i(\sigma^a\bar{\eta}')_\alpha \frac{\partial}{\partial y'^a} = -\frac{\partial}{\partial \tau^\alpha} + i(\sigma^a\bar{\tau})_\alpha \frac{\partial}{\partial u^a} \ , \\ \bar{\mathrm{D}}'_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\eta}'^{\dot{\alpha}}} = \frac{\partial}{\partial \bar{\tau}^{\dot{\alpha}}} - i(\tau\sigma^a)_{\dot{\alpha}} \frac{\partial}{\partial u^a} \ . \end{cases} \tag{6.7}
$$

Of course D_A and D'_A commute or anticommute with each other.

Under an infinitesimal deformation of the loop C, we obtain

$$
\begin{aligned}
&\delta \ln \mathbf{W}_S[C] \\
&= 2\oint e'^B \delta e'^A [\mathrm{D}'_A \oint e^C \mathcal{P}_{CB}(z',z) - (-)^{|A||B|}\mathrm{D}'_B \oint e^C \mathcal{P}_{CA}(z',z)] \\
&+4i\oint (e'^\alpha \delta e'^{\dot{\alpha}} + e'^{\dot{\alpha}}\delta e'^\alpha)\sigma^a_{\alpha\dot{\alpha}} \oint e^A \mathcal{P}_{Aa}(z',z) \ ,
\end{aligned} \tag{6.8}
$$

where we used (6.5) and $\delta e'^A$ is defined by

$$
\delta e'^A = \delta z^M(t) e_M{}^A(t) \, .
$$

The last term of the right hand side of (6.8) appears because we express it with the bases e^A. The additional anzatz we take for the solution reads

$$
\mathcal{P}_{A\dot{\alpha}} = 0
$$

in $\ln \mathbf{W}_S[C]$. This ansatz is consistent with the condition (2.12)

$$
0 = \bar{\mathrm{D}}_{\dot{\alpha}} \ln \mathbf{W}_S[C_{z_f z_i}] \ = 2\int_{z_i}^{z_f} e'^B \mathcal{P}_{B\dot{\alpha}}(z_f \text{ or } z_i,\, z') \, . \tag{6.9}
$$

Let us return to (6.8). The contour integral $\oint e'^B \delta e'^A$ in (6.8) represents an infinitesimal area element which has flat indices. Therefore we get, from (6.8), the following constraints given by (6.2)

$$
\begin{aligned}
0 \;&=\; \frac{\delta}{\delta \Sigma^{\alpha\beta}(z')} \ln \mathbf{W}_S[C] \\
&=\; 2\oint e^a \{\mathrm{D}'_\alpha \mathcal{P}_{a\beta}(z',z) + \mathrm{D}'_\beta \mathcal{P}_{a\alpha}(z',z)\} \\
&\quad -2\oint e^\gamma \{\mathrm{D}'_\alpha \mathcal{P}_{\gamma\beta}(z',z) + \mathrm{D}'_\beta \mathcal{P}_{\gamma\alpha}(z',z)\} \, ,
\end{aligned} \tag{6.10}
$$

$$
\begin{aligned}
0 &= \frac{\delta}{\delta\Sigma^{\dot\alpha\alpha}(z')}\ln\mathbf{W}_S[C] \\
&= 2\oint e^a\{\bar{\mathrm{D}}'_{\dot\alpha}\mathcal{P}_{a\alpha}(z',z)+2i\sigma^b_{\alpha\dot\alpha}\mathcal{P}_{ab}(z',z)\} \\
&\quad -2\oint e^\gamma\{\bar{\mathrm{D}}'_{\dot\alpha}\mathcal{P}_{\gamma\alpha}(z',z)+2i\sigma^b_{\alpha\dot\alpha}\mathcal{P}_{\gamma b}(z',z)\}\,, \qquad (6.11)
\end{aligned}
$$

where we used (6.5) and the condition $\mathcal{P}_{A\dot\alpha}=0$. The equation originating from the second one of (6.2) is trivially satisfied as $\mathcal{P}_{A\dot\alpha}=0$.

From (6.10), we obtain

$$
\begin{aligned}
0 &= \mathrm{D}'_\alpha\mathcal{P}_{a\beta}(z',z)+\mathrm{D}'_\beta\mathcal{P}_{a\alpha}(z',z)\,, \\
0 &= \mathrm{D}'_\alpha\mathcal{P}_{\gamma\beta}(z',z)+\mathrm{D}'_\beta\mathcal{P}_{\gamma\alpha}(z',z)\,, \qquad (6.12)
\end{aligned}
$$

and $\mathcal{P}_{a\alpha}$, $\mathcal{P}_{\gamma\alpha}$ are written as

$$
\mathcal{P}_{a\alpha}(z',z)=\mathrm{D}'_\alpha\mathcal{P}_a(z',z)\,,\ \mathcal{P}_{\gamma\alpha}(z',z)=\mathrm{D}'_\alpha\mathcal{P}_\gamma(z',z)\,. \qquad (6.13)
$$

for $\mathcal{P}_a(z',z)$, $\mathcal{P}_\gamma(z',z)$. From (6.11) and (6.13), we obtain

$$
\mathcal{P}_{ab}(z',z)=-\frac{i}{4}\bar{\mathrm{D}}'\bar\sigma_b\mathrm{D}'\mathcal{P}_a(z',z)\,,\ \mathcal{P}_{\gamma b}(z',z)=-\frac{i}{4}\bar{\mathrm{D}}'\bar\sigma_b\mathrm{D}'\mathcal{P}_\gamma(z',z)\,.
$$

Considering the above forms of $\mathcal{P}_{ab}$, $\mathcal{P}_{\gamma b}$ and also (6.5), we can express all $\mathcal{P}$'s with single function $\mathcal{P}(z',z)$:

$$
\begin{aligned}
\mathcal{P}_b(z',z) &= -\frac{i}{4}\bar{\mathrm{D}}\bar\sigma_b\mathrm{D}\mathcal{P}(z',z)\,, \\
\mathcal{P}_\gamma(z',z) &= \mathrm{D}_\gamma\mathcal{P}(z',z)\,, \\
\mathcal{P}_{ab}(z',z) &= (\frac{i}{4})^2\bar{\mathrm{D}}\bar\sigma_a\mathrm{D}\bar{\mathrm{D}}'\bar\sigma_b\mathrm{D}'\mathcal{P}(z',z)\,, \\
\mathcal{P}_{\gamma b}(z',z) &= -\frac{i}{4}\bar{\mathrm{D}}'\bar\sigma_b\mathrm{D}'\mathrm{D}_\gamma\mathcal{P}(z',z)\,, \\
\mathcal{P}_{\gamma\alpha}(z',z) &= \mathrm{D}'_\alpha\mathrm{D}_\gamma\mathcal{P}(z',z)\,. \qquad (6.14)
\end{aligned}
$$

Here $\mathcal{P}(z',z)$ must satisfy the condition

$$
\mathcal{P}(z',z)=\mathcal{P}(z,z')\,. \qquad (6.15)
$$

This function should be a function of u, τ, $\bar\tau$ only. We see that $\mathcal{P}$ must take the following form

$$
\mathcal{P}(z',z)=Q(u)+\tau^2\bar\tau^2R(u)\,. \qquad (6.16)
$$

The remaining task is to determine the function $\mathcal{P}(z,z')$ by considering equation (6.1). Using (6.8) we obtain

$$\begin{aligned} & 2\oint e^{a}\bar{\sigma}^{b\dot{\alpha}\alpha}\mathrm{D}'_{\alpha}\bar{\mathrm{D}}'_{\dot{\alpha}}\mathcal{P}_{ab}(z',z) \\ & +2\oint e^{\beta}\bar{\sigma}^{a\dot{\alpha}\alpha}\mathrm{D}'_{\alpha}\bar{\mathrm{D}}'_{\dot{\alpha}}\mathcal{P}_{\beta a}(z',z) \\ = \; & 8g^{2}\oint\{\mathcal{D}^{\alpha}_{f}\mathrm{D}_{\alpha}\delta(\tau)\delta(\bar{\tau})\delta^{4}(u)+e^{A}\mathrm{D}_{A}f(u,\tau,\bar{\tau})\}\,. \end{aligned} \tag{6.17}$$

Here we add the last term $8g^2\oint e^A\mathrm{D}_A f(u,\tau,\bar{\tau})$ to the right hand side of the Schwinger-Dyson equation. This term is auxiliary as it has the form of total derivatives for the contour integral. This additional term is not necessary and we add it only for convenience. The source term $\mathcal{D}^{\alpha}_{f}\mathrm{D}_{\alpha}\delta(\tau)\delta(\bar{\tau})\delta^{4}(u)$ does not involve the basis $e^{\dot{\alpha}}$. Following this fact, we impose the condition that f be chiral. The $f(u,\tau,\bar{\tau})$ must then have the form

$$f(u,\tau,\bar{\tau})=\mathrm{A}(u)+i\tau\sigma^{a}\bar{\tau}\partial_{a}\mathrm{A}(u)+\frac{1}{4}\tau^{2}\bar{\tau}^{2}\Box\mathrm{A}(u)+\tau^{2}\mathrm{F}(u)\,. \tag{6.18}$$

We now insert $\mathcal{P}_{ab}(z',z)$, and $\mathcal{P}_{\gamma b}(z',z)$ (see (6.13)) into (6.17). Using the concrete form of $\mathcal{P}$ given by (6.16) and matching appropriate power of τ, $\bar{\tau}$, we get the following independent differential equations of the fields $Q(u)$, $R(u)$ and the component fields of $f(u,\tau,\bar{\tau})$,

$$\begin{aligned} & \Box Q(u)-4R(u)=-4ig^{2}\mathrm{A}(u)\,, \\ & \Box(\Box Q(u)-4R(u))=4ig^{2}\delta^{4}(u)\,, \\ & \mathrm{F}(u)=0\,. \end{aligned} \tag{6.19}$$

Solving these equations, we find that

$$\begin{aligned} \mathcal{P}(z,z') \;=\; & 2ig^{2}\Box^{-1}(\Box^{-1}\delta^{4}(u)-C(u)) \\ & -\frac{i}{2}g^{2}\tau^{2}\bar{\tau}^{2}(\Box^{-1}\delta^{4}(u)+C(u))\,. \end{aligned} \tag{6.20}$$

Here the $C(u)$ is an arbitrary function and can be regarded as gauge parameter. We need not be concerned about the dependence of $\mathcal{P}(z,z')$ on $C(u)$. Only $\mathcal{P}_{AB}(z',z)$ are directly related to physical quantities and these are related to $\mathcal{P}(z,z')$ through the gauge independent combination $I(u)=\Box Q(u)-4R(u)$. Using the expressions (6.14), we obtain that

$$\begin{aligned} \mathcal{P}_{ab}(z',z) \;=\; & (\frac{i}{4})^{2}\{2g_{ab}-2i\tau\sigma_{a}\bar{\tau}\partial_{b}+2i\tau\sigma_{b}\bar{\tau}\partial_{a} \\ & +2\epsilon_{abcd}\partial^{c}\tau\sigma^{d}\bar{\tau}-\tau^{2}\bar{\tau}^{2}(g_{ab}\Box-\partial_{a}\partial_{b})\}I(u)\,, \\ \mathcal{P}_{\alpha b}(z',z) \;=\; & -\frac{i}{4}\{(\sigma_{a}\bar{\tau})_{\alpha}+\frac{i}{2}\bar{\tau}^{2}(\sigma^{b}\bar{\sigma}_{a}\tau)_{\alpha}\partial_{b} \end{aligned}$$

$$
\begin{aligned}
& \quad +i\tau_\alpha \bar{\tau}^2 \partial_a \} I(u) \,, \\
\mathcal{P}_{\alpha\beta}(z', z) &= \frac{1}{2} \bar{\tau}^2 \epsilon_{\alpha\beta} I(u) \,, \\
I(u) &= 4ig^2 \Box^{-1} \delta^4(u) \,. \qquad (6.21)
\end{aligned}
$$

These results are equal to those obtained by carrying out the path integral for the super Wilson-loop average. So we can conclude that our Schwinger-Dyson equation is nontrivial and provides no less information than the path integral method does. We should note that the equations given by (6.2) are as important as (6.1) in order to solve our Schwinger-Dyson equation.

VII. One-dimensional fermion along the loop and renormalization of the one-point super Wilson-loop average

In the previous section, we obtained the abelian as well as the nonabelian supersymmetric Schwinger-Dyson equations. These are, however, written in terms of bare quantities and must be renormalized in order to obtain physical results. For the ordinary non-supersymmetric Wilson-loop, the renormalization of the one-point function has been already discussed in refs. [13]-[17]. The renormalization of the Wilson-loop is most transparent in terms of the first quantized lagrangian minimally coupled to the gauge fields (see [12, 13]):

$$
\mathcal{L}_{path} = \int_0^1 ds \, [i\bar{w}(s)\dot{w}(s) + g\bar{w}(s)\dot{x}^a(s)v_a(x(s))w(s)] \,, \qquad (7.1)
$$

where $w(s)$ is a one-dimensional fermion on a given path $x^a(s)$ and belongs to the fundamental representation of $SU(N_c)$ or $U(N_c)$. In this language, the path-ordered exponential is represented as the two point function

$$
W[C] = [P \exp\{ig \int_0^1 ds\dot{x}v_a\}]^{ij} = \langle 0| P \, w^i(0)\bar{w}^j(1)|0\rangle \,. \qquad (7.2)
$$

The proof of this equality becomes obvious if one recognizes that the right hand side of the above equation is a Green's function $G(s=1)$ satisfying

$$
\left(i\frac{d}{ds} + g\dot{x}^a(s)v_a(s) \right) G(s) = i\delta(s)\delta_i^j \quad . \qquad (7.3)
$$

The solution of this equation under $G(0) = 1$ is in fact the left hand side of (7.2) with multiplication of the step function $\theta(s)$ implied. Therefore, the renormalization problem of the Wilson-loop - a composite operator - becomes equivalent to that of the w-field in the lagrangian $\mathcal{L}_{path} + \mathcal{L}_{YM}$

We may expect that the renormalization problem of our super Wilson-loop can be handled in a similar way to the non-supersymmetric case. Let us define

$$\mathcal{L}_{path} = \int_0^1 ds\,[i\bar{w}(s)\dot{w}(s) - i\bar{w}(s)\mathcal{A}(z(s))w(s)]\,, \tag{7.4}$$

where $\mathcal{A}(z) = e^A\mathcal{A}_A$ is the one-form in superspace. Using $\mathcal{L}_{path}$, we can write the super path-ordered exponential in the form similar to (7.2),

$$W_S[C] = [P\exp\oint\mathcal{A}(z(s))]^{ij} = \langle 0|P\,w^i(0)\bar{w}^j(1)|0\rangle\,. \tag{7.5}$$

Let us demonstrate the renormalization of the self-energy part of the w-field to one-loop order by using the dimensional regularization. We will show that the pole term appearing in the self-energy part is cancelled by the local counterterm

$$\int_0^1 ds\,i(Z_2 - 1)\bar{w}(s)\dot{w}(s)\,. \tag{7.6}$$

Here Z_2 denotes the renormalization constant of the wave function for the w field.

At one-loop, all we have to do is to consider the leading of the vertex $\int\bar{w}\mathcal{A}w$ expanded in V. The ghost loop does not contribute to the one-loop self-energy part. Knowing these we can proceed without taking the Wess-Zumino gauge. The propagator for the w-field and that for the superfield $\mathrm{V}(z)$ are respectively

$$\begin{aligned}
&\langle w^{(i)}(s_1)\bar{w}^{(j)}(s_2)\rangle = \delta^{ij}\theta(s_2 - s_1)\\
&\langle \mathrm{V}^{(r)}(z)V^{(s)}(z')\rangle\\
&= -ig^2\delta^{rs}\{-4(1-\alpha)\Box_{\mathrm{D}}^{-2}(u) + (1+\alpha)\tau^2\bar{\tau}^2\Box_{\mathrm{D}}^{-1}(u)\}\\
&\equiv -ig^2\delta^{rs}\mathrm{V}(z,z')\,,
\end{aligned} \tag{7.7}$$

where $\Box_{\mathrm{D}}^{-1}(u)$ and $\Box_{\mathrm{D}}^{-2}(u)$ are defined respectively by the following forms

$$\begin{aligned}
\Box_{\mathrm{D}}^{-1}(u) &= \int\frac{d^{\mathrm{D}}k}{(2\pi)^{\mathrm{D}}}\frac{\mathrm{e}^{-iku}}{-k^2} = -\frac{i}{4\pi^{\mathrm{D}/2}}\frac{\Gamma(\frac{\mathrm{D}}{2}-1)}{u^{\mathrm{D}-2}}\,,\\
\Box_{\mathrm{D}}^{-2}(u) &= \int\frac{d^{\mathrm{D}}k}{(2\pi)^{\mathrm{D}}}\frac{\mathrm{e}^{-iku}}{(-k^2)^2} = \frac{i}{16\pi^{\mathrm{D}/2}}\frac{\Gamma(\frac{\mathrm{D}}{2}-2)}{u^{\mathrm{D}-4}}\,.
\end{aligned} \tag{7.8}$$

The variables u, τ and $\bar{\tau}$ are the same ones as defined by (6.6). Here u is defined by

$$u = \sqrt{u^a u_a} = \sqrt{-u_0^2 + u_1^2 + u_2^2 + u_3^2}\,. \tag{7.9}$$

Following this definition of u, the Lorentz index a runs only from 0 to 3. The matrices σ^a or $\bar{\sigma}^a$ are defined in 4-dimensional spacetime.

$\Box_{\mathrm{D}}^{-1}(u)$ and $\Box_{\mathrm{D}}^{-2}(u)$ satisfy

$$\begin{aligned}
\Box\Box_{\mathrm{D}}^{-2}(u) &= (1+\epsilon)\Box_{\mathrm{D}}^{-1}(u)\,, \\
\Box\Box_{\mathrm{D}}^{-1}(u) &= \delta_{\mathrm{D}}(u)\,, \\
\delta_{\mathrm{D}}(u) &= i\frac{\Gamma(\mathrm{D}/2)}{\pi^{\mathrm{D}/2}}\epsilon u^{2\epsilon-4}\,, \\
\epsilon &= \frac{4-\mathrm{D}}{2}\,,
\end{aligned}$$

where $\Box$ denotes the four dimensional Klein-Gordon operator.

The propagator for V is invariant under the four dimensional supersymmetry transformations given by (5.16) and is expressible in terms of u, τ and $\bar{\tau}$. Using this propagator and the vertex interaction $\int ds\,\bar{w}\mathcal{A}w$ in $\mathcal{L}_{Seff}$, we find

$$\begin{aligned}
i\Sigma^{(1)} &= (i)^2\int_0^1 ds_1\int_0^1 ds_2\bar{w}(s_1)T^rT^s w(s_2)\theta(s_2-s_1)\langle\mathcal{A}^{(r)}(z)\mathcal{A}^{(s)}(z')\rangle \\
&= -ig^2(i)^2C_2(N_c)\int_0^1 ds_1\int_0^1 ds_2\bar{w}(s_1)w(s_2)\theta(s_2-s_1) \\
&\quad\times\{(\frac{i}{4})^2e^ae'^b\bar{\mathrm{D}}\bar{\sigma}_a\mathrm{D}\,\bar{\mathrm{D}}'\bar{\sigma}_b\mathrm{D}' \\
&\quad-\frac{i}{4}e^ae'^\alpha\bar{\mathrm{D}}\bar{\sigma}_a\mathrm{D}\,\mathrm{D}'_\alpha-\frac{i}{4}e^\alpha e'^a\mathrm{D}_\alpha\,\bar{\mathrm{D}}'\bar{\sigma}_a\mathrm{D}' \\
&\quad+e^\alpha e'^\beta\mathrm{D}_\alpha\mathrm{D}'_\beta\}\mathrm{V}(z,z')\,, \\
C_2(N_c) &= \frac{N_c^2-1}{N_c}\ \text{for gauge group } SU(N_c)\,,\ N_c,\ \text{for gauge group } U(N_c)\,, \qquad (7.10)
\end{aligned}$$

where we have taken into account that only the leading order in V in the vertex contributes to the one-loop self energy as we mentioned above.

Let us evaluate the first term in $i\Sigma^{(1)}$, which we denote by iI_1. The term iI_1 is given by

$$\begin{aligned}
iI_1 &\equiv -i(\frac{i}{4})^2(ig)^2C_2(N_c)\int_0^1 ds_1\int_{s_1}^1 ds_2\,\bar{w}(s_1)w(s_2)e^ae'^b \\
&\quad\times\{-16g_{ab}\Box_{\mathrm{D}}^{-1}(u)+16i\tau\sigma_b\bar{\tau}\partial_a\Box_{\mathrm{D}}^{-1}(u)-16i\tau\sigma_a\bar{\tau}\partial_b\Box_{\mathrm{D}}^{-1}(u) \\
&\quad-16\epsilon_{abcd}\tau\sigma^c\bar{\tau}\partial^d\Box_{\mathrm{D}}^{-1}(u)+4\tau^2\bar{\tau}^2g_{ab}\delta_{\mathrm{D}}(u)\}k(\epsilon,\alpha)\,, \\
&\quad k(\epsilon,\alpha)=1+\frac{1}{2}\epsilon(1-\alpha)\,,\ \epsilon=\frac{4-\mathrm{D}}{2}\,. \qquad (7.11)
\end{aligned}$$

We have used (7.7). The function $\delta_{\mathrm{D}}(u)$ is a well-defined distribution and does not have poles at $\mathrm{D}=4$. The last term in the middle bracket does not contribute to the divergence. The

divergence comes from the point $s_1 = s_2$. In order to estimate the behavior of the integrand of (7.11) around this point, we expand as Taylor series

$$
\begin{aligned}
&e'^a = e^a(s_2) = e^a(s_1) + (s_2 - s_1)de^a(s_1) + \cdots , \\
&w(s_2) = w(s_1) + (s_2 - s_1)\dot{w}(s_1) + \cdots , \\
&u^a = (s_1 - s_2)e^a(s_1) - \frac{1}{2}(s_1 - s_2)^2 de^a(s_1) + \cdots , \\
&\tau = (s_1 - s_2)\dot{\eta}(s_1) + \cdots , \\
&\bar{\tau} = (s_1 - s_2)\dot{\bar{\eta}}(s_1) + \cdots .
\end{aligned} \tag{7.12}
$$

The first term in I_1 reads

$$
\begin{aligned}
&-i(\frac{i}{4})^2(ig)^2 C_2(N_c)\int_0^1 ds_1 \int_{s_1}^1 ds_2 \bar{w}(s_1)w(s_2)e^a e'^b(-16)g_{ab}\Box_{\mathrm{D}}^{-1}(u) \\
= \;& 16(\frac{i}{4})^2(ig)^2 C_2(N_c)\frac{1}{4\pi^{\mathrm{D}/2}}\Gamma(\frac{\mathrm{D}}{2}-1)\int_0^1 ds_1 \int_{s_1}^1 ds_2\, |e|^{4-\mathrm{D}}\{(s_1 - s_2)^{2-\mathrm{D}}\bar{w}(s_1)w(s_1) \\
&+\frac{1}{2}\frac{ede}{|e|^2}(\mathrm{D}-4)(s_1 - s_2)^{3-\mathrm{D}}\bar{w}(s_1)w(s_1) + (s_1 - s_2)^{3-\mathrm{D}}\bar{w}(s_1)\dot{w}(s_1)\}k(\epsilon,\alpha) \\
&+O((\mathrm{D}-4))\,, \qquad\qquad |e|^2 = e^a e_a\,.
\end{aligned} \tag{7.13}
$$

The term of $(s_1 - s_2)^{2-\mathrm{D}}$ does not yield any pole at $\mathrm{D} = 4$ after integrating over s_2. The second term yields a pole, but it gets multiplied by the factor $(\mathrm{D} - 4)$. The contribution to the pole at $\mathrm{D} = 4$ comes only from the last term of (7.13). We find

$$
\begin{aligned}
&\frac{1}{8\pi^2}(ig)^2 C_2(N_c)\int_0^1 ds_1\, \bar{w}(s_1)\dot{w}(s_1)\{-\frac{2}{4-\mathrm{D}} - \gamma - \frac{1}{2}(1-\alpha) - \ln(\pi|e|^2(1-s_1)^2)\} \\
&+(\text{finite term at } \mathrm{D} = 4)\,.
\end{aligned} \tag{7.14}
$$

Next we evaluate the second, third and forth terms in the middle bracket of the integrand of (7.11). By using (7.12), we see that each of these terms has poles but does not contribute to the self-energy part because of antisymmetry with respect to the indices a and b.

We will estimate the second and the third terms in $i\Sigma^{(1)}$, which we denote respectively by iI_2 and iI_3. Using the expansion (7.12), the second term iI_2 is evaluated as

$$
\begin{aligned}
iI_2 \;\equiv\;& i\frac{i}{4}(ig)^2 C_2(N_c)\int_0^1 ds_1 \int_0^1 ds_2\, \bar{w}(s_1)w(s_2)\theta(s_2 - s_1)e^a e'^\alpha \bar{\mathrm{D}}\bar{\sigma}_a \mathrm{D}\, \mathrm{D}'_\alpha V(z,z') \\
=\;& i\frac{i}{4}(ig)^2 C_2(N_c)\int_0^1 ds_1 \int_{s_1}^1 ds_2\, \bar{w}(s_1)w(s_2)e^a e'^\alpha\{8(\sigma_a\bar{\tau})_\alpha \Box_{\mathrm{D}}^{-1} \\
&-8i\tau_\alpha\bar{\tau}^2\partial_a\Box_{\mathrm{D}}^{-1} - 4i\bar{\tau}^2(\sigma^b\bar{\sigma}_a\tau)_\alpha\partial_b\Box_{\mathrm{D}}^{-1}\}k(\epsilon,\alpha) \\
=\;& -\frac{i}{2\pi^{\mathrm{D}/2}}(ig)^2 C_2(N_c)\int_0^1 ds_1\, \bar{w}(s_1)w(s_1)|e|^{2-\mathrm{D}}\Gamma(\frac{\mathrm{D}}{2}-1)\frac{1}{4-\mathrm{D}}e^a e^\alpha \sigma_{a\alpha\dot{\alpha}}e^{\dot{\alpha}}(s_1 - 1)^{4-\mathrm{D}} \\
&+(\text{finite term at } \mathrm{D} = 4)\,.
\end{aligned} \tag{7.15}
$$

Similarly, the third term iI_3 is evaluated as

$$\begin{aligned} iI_3 &\equiv i\frac{i}{4}(ig)^2 C_2(N_c)\int_0^1 ds_1 \int_0^1 ds_2\, \bar{w}(s_1)w(s_2)\theta(s_2-s_1)e'^{a}e^{\alpha}\bar{\mathrm{D}}'\bar{\sigma}_a\mathrm{D}'\,\mathrm{D}_\alpha V(z,z') \\ &= \frac{i}{2\pi^{\mathrm{D}/2}}(ig)^2 C_2(N_c)\int_0^1 ds_1\, \bar{w}(s_1)w(s_1)|e|^{2-\mathrm{D}}\Gamma(\frac{\mathrm{D}}{2}-1)\frac{1}{4-\mathrm{D}}e^a e^\alpha \sigma_{a\alpha\dot{\alpha}}e^{\dot{\alpha}}(s_1-1)^{4-\mathrm{D}} \\ &\quad +(\text{finite term at D}=4)\,. \end{aligned} \tag{7.16}$$

The first line of iI_3 has an opposite sign to that of iI_2. We conclude that $iI_2 + iI_3$ does not have a pole at $\mathrm{D}=4$. The integrand of the last term in $i\Sigma^{(1)}$ includes only higher powers of $(s_1 - s_2)$ than $4-\mathrm{D}$ when expanded around s_1. It does not yield a pole.

Putting all these together, we obtain

$$\begin{aligned} i\Sigma^{(1)} &= \frac{1}{8\pi^2}(ig)^2 C_2(N_c)\int_0^1 ds_1\, \bar{w}(s_1)\dot{w}(s_1)\{-\frac{2}{4-\mathrm{D}}+\gamma-\frac{1}{2}(1-\alpha)-\ln(\pi|e|^2(1-s_1)^2)\} \\ &\quad +(\text{finite term at D}=4)\,. \end{aligned} \tag{7.17}$$

At one-loop level, the self-energy part of the w-field is renormalized by the local counterterm

$$\int_0^1 ds\, i(Z_2-1)\bar{w}(s)\dot{w}(s)\,,$$

with Z_2 given by

$$Z_2 = 1 - \frac{g^2}{8\pi^2}C_2(N_c)\{-\frac{2}{4-\mathrm{D}}+\gamma-\frac{1}{2}(1-\alpha)\}\,. \tag{7.18}$$

This one-loop renormalization constant Z_2 is the same one as in the non-supersymmetric case.

VIII. Discussion

We have already summarized the results from our investigation in introduction. Let us briefly discuss a few points which we would like to pursue. In section seven, we considered the renormalization of one-point function for our super Wilson-loop and carried out the explicit one-loop computation. We should, however, consider the renormalization of our Schwinger-Dyson equation (5.15) as well. One may expect that this could be done by smearing the delta function $\delta(z-z')$ appearing in (5.15) by a heat kernel. Another obvious direction is extension of our work to the $N=2$ case. We have made some preliminary investigations and hope to be able to report on this. A wealth of results will be waiting on this avenue in connection with [18, 19].

Appendix A. Wilson-loop and area derivative

We represent the gauge field by $v_a(x) = \sum_r v_a^{(r)}(x)T^{(r)}$ where $T^{(r)}$ are the generators of the gauge group. Using the spacetime curve C_{xy}, we define the Wilson-loop as follows,

$$W[C_{xy}] = P\exp\{-\frac{i}{2}\int_y^x dw^a v_a(w)\} \ , \tag{A.1}$$

where the capital P denotes the usual path ordered product. The field strength in this notation reads

$$v_{ab} = \partial_a v_b - \partial_b v_a + \frac{i}{2}[v_a, v_b] \ .$$

We simply list

$$\begin{aligned} \frac{\partial}{\partial y^a}W[C_{xy}] &= +\frac{i}{2}W[C_{xy}]v_a(y)\,, \\ \frac{\partial}{\partial x^a}W[C_{xy}] &= -\frac{i}{2}v_a(x)W[C_{xy}]\,. \end{aligned} \tag{A.2}$$

Let us now consider two parameters s, t, and a function $u^a(s,t)$ which maps these parameters into spacetime. For convenience we take the condition $u^a(0,0) = x^a$. We consider two small curves $\triangle C_1$ and $\triangle C_2$ defined by $u^a(s,t)$ as follows:

$$\triangle C_1 \ : \ u(0,0) \xrightarrow{\text{fixing}\, t=0} u(\delta s, 0) \xrightarrow{\text{fixing}\, s=\delta s} u(\delta s, \delta t)\,, \tag{A.3}$$

$$\triangle C_2 \ : \ u(0,0) \xrightarrow{\text{fixing}\, s=0} u(0, \delta t) \xrightarrow{\text{fixing}\, t=\delta t} u(\delta s, \delta t)\,, \tag{A.4}$$

where δs and δt are infinitesimal. Now we will evaluate the difference between $W[\triangle C_1 + C_{xy}]$ and $W[\triangle C_2 + C_{xy}]$,

$$\begin{aligned} \delta W[C_{xy}] &\equiv W[\triangle C_1 + C_{xy}] - W[\triangle C_2 + C_{xy}] \\ &= \{W[\triangle C_1] - W[\triangle C_2]\}W[C_{xy}]\,, \end{aligned} \tag{A.5}$$

where $\delta W[C_{xy}]$ denotes that difference. Keeping the term up to the second order in δs and δt, we get

$$\begin{aligned} W[\triangle C_1] - W[\triangle C_2] &= -\frac{i}{2}P\{\int_{\triangle C_1} - \int_{\triangle C_2}\}dw^a v_a(w) \\ &\quad +\frac{1}{2}(-\frac{i}{2})^2 P\{(\int_{\triangle C_1} dw^a v_a(w))^2 - (\int_{\triangle C_2} dw^a v_a(w))^2\} \\ &\quad +O(\delta^3)\,. \end{aligned} \tag{A.6}$$

Calculating $\int_{\triangle C_1} dw^a v_a(w)$ of the right hand side of (A.6), we find that

$$\begin{aligned}\int_{\triangle C_1} dw^a v_a(w) &= \{\delta s\frac{\partial u^a}{\partial s}+\delta t\frac{\partial u^a}{\partial t}\}v_a(x)\\ &+ \frac{1}{2}(\delta s)^2\{\frac{\partial^2 u^a}{\partial s^2}v_a(x)+\frac{\partial u^a}{\partial s}\frac{\partial u^b}{\partial s}\partial_a v_b(x)\}\\ &+ \frac{1}{2}(\delta t)^2\{\frac{\partial^2 u^a}{\partial t^2}v_a(x)+\frac{\partial u^a}{\partial t}\frac{\partial u^b}{\partial t}\partial_a v_b(x)\}\\ &+ \delta s\delta t \quad \{\frac{\partial^2 u^a}{\partial s\partial t}v_a(x)+\frac{\partial u^a}{\partial s}\frac{\partial u^b}{\partial t}\partial_a v_b(x)\}\ ,\end{aligned} \tag{A.7}$$

and $\int_{\triangle C_2} dw^a v_a(w)$ is given similarly as $\int_{\triangle C_1} dw^a v_a(w)$ with δs and δt exchanged. So we obtain

$$-\frac{i}{2}P\{\int_{\triangle C_1}-\int_{\triangle C_2}\}dw^a v_a(w) = -\frac{i}{2}\delta s\delta t\frac{1}{2}\{\frac{\partial u^a}{\partial s}\frac{\partial u^b}{\partial t}-\frac{\partial u^b}{\partial s}\frac{\partial u^a}{\partial t}\}(\partial_a v_b(x)-\partial_b v_a(x))\,. \tag{A.8}$$

Calculating $P(\int_{\triangle C_1} dw^a v_a(w))^2$ of the right hand side of (A.6), we get

$$\begin{aligned}P(\int_{\triangle C_1} dw^a v_a(w))^2 &= \delta s^2\{\frac{\partial u^a}{\partial s}v_a(x)\}^2+\delta t^2\{\frac{\partial u^a}{\partial t}v_a(x)\}^2\\ &+ 2\delta s\delta t\frac{\partial u^a}{\partial s}\frac{\partial u^b}{\partial t}v_b(x)v_a(x)\ ,\end{aligned} \tag{A.9}$$

and $P(\int_{\triangle C_2} dw^a v_a(w))^2$ is the same as $P(\int_{\triangle C_1} dw^a v_a(w))^2$ with δs and δt exchanged. So we obtain

$$\begin{aligned}&\frac{1}{2}(-\frac{i}{2})^2P\{(\int_{\triangle C_1} dw^a v_a(w))^2-(\int_{\triangle C_2} dw^a v_a(w))^2\}\\ &= -\frac{1}{2}(-\frac{i}{2})^2\delta s\delta t\{\frac{\partial u^a}{\partial s}\frac{\partial u^b}{\partial t}-\frac{\partial u^b}{\partial s}\frac{\partial u^a}{\partial t}\}[v_a(x),v_b(x)]\,.\end{aligned} \tag{A.10}$$

From the equation (A.8) and (A.10), we see that

$$\delta W[C_{xy}] = (-\frac{i}{2})\frac{1}{2}\delta s\delta t\{\frac{\partial u^a}{\partial s}\frac{\partial u^b}{\partial t}-\frac{\partial u^b}{\partial s}\frac{\partial u^a}{\partial t}\}v_{ab}(x)W[C_{xy}]\,. \tag{A.11}$$

Let us consider a closed curve $\triangle C \equiv \triangle C_1 - \triangle C_2$. We project $\triangle C$ to the (x^a, x^b)-plane and express this curve $\triangle C^{ab}$. $\frac{1}{2}\delta s\delta t\{\frac{\partial u^a}{\partial s}\frac{\partial u^b}{\partial t}-\frac{\partial u^b}{\partial s}\frac{\partial u^a}{\partial t}\}$ is the area which is surrounded by the small curve $\triangle C^{ab}$. We define the area element $\delta\sigma^{ab}$ as this small area and we obtain the area derivative of the Wilson-loop as follows:

$$\begin{aligned}\frac{\delta W[C_{xy}]}{\delta\sigma^{ab}(x)} &= -\frac{i}{2}v_{ab}(x)W[C_{xy}]\,,\\ \delta\sigma^{ab} &= \frac{1}{2}\delta s\delta t\{\frac{\partial u^a}{\partial s}\frac{\partial u^b}{\partial t}-\frac{\partial u^b}{\partial s}\frac{\partial u^a}{\partial t}\}\ .\end{aligned} \tag{A.12}$$

The gauge transformation of the Wilson-loop takes the form;

$$W[C_{xy}] \longrightarrow W'[C_{xy}] = U(x)W[C_{xy}]U^{-1}(y)\,. \tag{A.13}$$

In the case that the curve C_{xy} is a closed loop C_{xx}, we see from (A.13) that the operator $trW[C_{xx}]$ is gauge invariant. Since $trW[C_{xx}]$ doesn't depend on x, we may express this as $trW[C]$. Note that, if we don't take the trace, $W[C_{xx}]$ depends on x. Note also that $W[C_{xx}]$ is a matrix and gauge variant operator although C_{xx} is a closed loop.

Appendix B. Migdal-Makeenko equation

Let us start with

$$\begin{aligned} 0 &= \langle tr\, T^{(r)} \frac{\delta}{\delta v_a^{(r)}(x)} W[C] \rangle \\ &= \int [dv^a]\, tr(\, T^{(r)} \frac{\delta}{\delta v_a^{(r)}(x)} \exp\{i\frac{1}{g^2}\mathrm{S}_{YM}\} W[C]\,)\,, \end{aligned} \tag{B.1}$$

where the summation over the index r is taken tacitly. Here S_{YM} denotes the Yang-Mills action which is written as follows:

$$\begin{aligned} \mathrm{S}_{YM} &= \int d^4x \mathcal{L}_{YM}\,, \\ \mathcal{L}_{YM} &= tr\{-\frac{1}{16} v^{ab} v_{ab}\}\,, \end{aligned} \tag{B.2}$$

where $\mathcal{L}_{YM}$ denotes the Yang-Mills Lagrangian density. The path integral volume element $[dv^a]$ is normalized to satisfy the condition; $1 = \int [dv^a] \exp\{i\frac{1}{g^2}\mathrm{S}_{YM}\}$. The equation (B.1) is trivial, because it is the total derivative of the functional variable v^a. Letting the functional derivative $T^{(r)} \frac{\delta}{\delta v_a^{(r)}(x)}$ act on $\exp\{i\mathrm{S}_{YM}\}$, we find

$$T^{(r)} \frac{\delta}{\delta v_a^{(r)}(x)} \exp\{i\frac{1}{g^2}\mathrm{S}_{YM}\} = i\frac{1}{4g^2} D_b v^{ba}(x) \exp\{i\frac{1}{g^2}\mathrm{S}_{YM}\}\ , \tag{B.3}$$

where we have defined $D_b v^{ba}$ by

$$D_b v^{ba}(x) = \partial_b v^{ba}(x) + \frac{i}{2}\,[v_b(x), v^{ba}(x)]\,. \tag{B.4}$$

eq. (B.1) gives

$$i\frac{1}{4g^2}\langle tr\{\, D_b v^{ba}(x) W[C_{xx}]\}\rangle \;=\; \frac{i}{2}\oint dw^a \langle tr\{\, T^{(r)} W[C_{xw}] T^{(r)} \delta^{(4)}(w-x) W[C_{wx}]\,\}\rangle\,. \tag{B.5}$$

Here the $\oint$ denotes the contour integral along the closed loop C.

Let us try to rewrite the equation (B.5). Remember that the field strength v^{ba} is given by the area derivative of the Wilson-loop defined in (A.12), and the gauge connection v_b by the derivative of the Wilson-loop defined in (A.2). From the equation (A.12), we obtain

$$\frac{\delta}{\delta\sigma_{ba}(x)}\langle trW[C]\rangle = -\frac{i}{2}\langle tr\{v^{ba}(x)W[C_{xx}]\}\rangle\,. \tag{B.6}$$

Taking the derivative of the each side of the equation (B.6), yields

$$\begin{aligned}\frac{\partial}{\partial x^b}\frac{\delta}{\delta\sigma_{ba}(x)}\langle trW[C]\rangle &= -\frac{i}{2}\langle tr\{\Big(\frac{\partial}{\partial x^b}v^{ba}(x)\Big)W[C_{xx}]\}\rangle \\ &\quad -\frac{i}{2}\langle tr\{v^{ba}(x)\Big(\frac{\partial}{\partial x^b}W[C_{xx}]\}\Big)\rangle\,.\end{aligned} \tag{B.7}$$

Recall also that $W[C_{xx}]$ depends on x. From the equation (A.2) we see that

$$\frac{\partial}{\partial x^b}W[C_{xx}] = -\frac{i}{2}[v_b(x), W[C_{xx}]]\,. \tag{B.8}$$

Inserting (B.8) into (B.7), we find

$$\frac{\partial}{\partial x^b}\frac{\delta}{\delta\sigma_{ba}(x)}\langle trW[C]\rangle = -\frac{i}{2}\langle tr\{\,D_b v^{ba}(x)W[C_{xx}]\}\rangle\,. \tag{B.9}$$

This tells us that we may express the equation (B.5) as follows:

$$\begin{aligned}&-\frac{1}{2g^2}\frac{\partial}{\partial x^b}\frac{\delta}{\delta\sigma_{ba}(x)}\langle trW[C]\rangle \\ &= \frac{i}{2}\oint dw^a\langle tr\{\,T^{(r)}W[C_{xw}]T^{(r)}\delta^{(4)}(w-x)W[C_{wx}]\,\}\rangle\,.\end{aligned} \tag{B.10}$$

Using the property for the fundamental representation of $U(N_c)$

$$\sum_r T^{(r)}_{ij}\,T^{(r)}_{kl} = \delta_{il}\delta_{jk}\,, \tag{B.11}$$

the equation (B.10) yields

$$\begin{aligned}&-\frac{1}{2g^2}\frac{\partial}{\partial x^b}\frac{\delta}{\delta\sigma_{ba}(x)}\langle trW[C]\rangle \\ &= \frac{i}{2}\oint dw^a\delta^{(4)}(w-x)\langle tr\{\,W[C_{xw}]\,\}tr\{\,W[C_{wx}]\,\}\rangle\,,\end{aligned} \tag{B.12}$$

and this is the exact Schwinger-Dyson equation of the Wilson-loop for the gauge group $U(N_c)$. (B.12) is invariant under the gauge transformation. The left hand side of (B.12) is trivially invariant because of the invariance of $trW[C]$. As for the right hand side, because

of the delta function, the integrand is non-vanishing only if $w = x$ and, when $w = x$, each of $trW[C_{xw}]$ and $trW[C_{xw}]$ is gauge invariant.Therefore we see that (B.12) is gauge invariant.

Here we define the new coupling constant $g_c^2 \equiv N_c g^2$ and use g_c instead of g in the equation (B.12). Taking the large N_c limit with g_c kept finite, in the leading order of N_c we obtain the final form;

$$\frac{1}{g_c^2}\frac{\partial}{\partial x^b}\frac{\delta}{\delta\sigma_{ba}(x)}\mathbf{W}[C] \;=\; -i\oint dw^a \delta^{(4)}(w-x)\mathbf{W}[C_{xw}]\mathbf{W}[C_{wx}], \tag{B.13}$$

where $\mathbf{W}[C_{xy}]$ denotes the quantum average of the Wilson-loop;

$$\mathbf{W}[C_{xy}] = \langle\frac{1}{N_c}tr\,W[C_{xy}]\rangle = \int[dv^a]\exp\{i\frac{N_c}{g_c^2}\mathrm{S}_{YM}\}\frac{1}{N_c}tr\,W[C_{xy}]\,.$$

The non-linear equation (B.13) for the $\mathbf{W}[C]$ is well known as Migdal-Makeenko equation. Note that in the leading order of N_c, $\mathbf{W}[C_{xy}]$ is composed of only the planar diagrams.

In the abelian case we can easily obtain the Schwinger-Dyson equation which corresponds to the equation (B.13);

$$\frac{1}{g^2}\frac{\partial}{\partial x^b}\frac{\delta}{\delta\sigma_{ba}(x)}\mathbf{W}[C] \;=\; -i\,\mathbf{W}[C]\oint dw^a \delta^{(4)}(w-x)\;. \tag{B.14}$$

In this case we can calculate $\mathbf{W}[C]$ directly from its definition;

$$\mathbf{W}[C] = \langle W[C]\rangle = \int[dv^a]\exp\{i\frac{1}{g^2}\mathrm{S}_{AB}\}W[C]\,,$$

where S_{AB} is the abelian action. We can carry out this path integral because of its Gaussian type. The result is

$$\begin{aligned}\mathbf{W}[C] &= \exp\{-\frac{i}{2}g^2\oint\oint dx^a dy_a \Box^{-1}(x-y)\}\\ &= \exp\{-\frac{i}{2}g^2\frac{i}{4\pi^2}\oint\oint\frac{dx^a dy_a}{(x-y)^2}\}\,.\end{aligned} \tag{B.15}$$

This satisfies the equation (B.14).

Appendix C. Vector Superfields

The elements of superspace are denoted by

$$z^M = (x^a, \theta^\alpha, \bar\theta^{\dot\alpha})\,, \tag{C.1}$$

where the capital letter M represents the four-vector index a as well as the spinor indices α and $\dot{\alpha}$. Elements of superspace obey the following multiplication law:

$$z^M z^N = (-)^{|M||N|} z^N z^M \,.$$

Here $|M|$ is the following function of M:

$$|M| = \begin{cases} 0, \text{ when } M \text{ is a vector index,} \\ 1, \text{ when } M \text{ is a spinor index.} \end{cases}$$

Exterior products in superspace are defined in complete analogy to ordinary space:

$$\begin{aligned} dz^M \wedge dz^N &= -(-)^{|M||N|} dz^N \wedge dz^M \,, \\ dz^M z^N &= (-)^{|M||N|} z^N dz^M \,. \end{aligned}$$

The p-forms on superspace is expressed as

$$\Omega_{\rm p} = dz^{M_1} \wedge dz^{M_2} \wedge \cdots \wedge dz^{M_{\rm p}} \Omega_{M_{\rm p} \cdots M_2 M_1}(z), \tag{C.2}$$

where we implicitly take the summation of $M_1, M_2, \cdots, M_{\rm p}$ as in the usual manner for upper and lower indices. From now on we shall drop the symbol $\wedge$ for exterior multiplication.

We now introduce exterior derivatives on superspace which map p-forms into (p+1)-forms,

$$\begin{aligned} \Omega_{\rm p} &= dz^{M_1} \cdots dz^{M_{\rm p}} \Omega_{M_{\rm p} \cdots M_1}(z) \,, \\ d\Omega_{\rm p} &= dz^{M_1} \cdots dz^{M_{\rm p}} dz^M \frac{\partial}{\partial z^M} \Omega_{M_{\rm p} \cdots M_1}(z) \,. \end{aligned} \tag{C.3}$$

From this definition of exterior derivatives, we can straightforwardly show the following properties:

$$\begin{aligned} d(\Omega_{\rm p} + \Sigma_{\rm p}) &= d\Omega_{\rm p} + d\Sigma_{\rm p} \,, \\ d(\Omega_{\rm p} \Omega_{\rm q}) &= \Omega_{\rm p} d\Omega_{\rm q} + (-)^{\rm q} (d\Omega_{\rm p}) \Omega_{\rm q} \,, \\ dd &= 0 \,, \end{aligned} \tag{C.4}$$

where $\Omega_{\rm p}$, $\Sigma_{\rm p}$ are p-forms and $\Omega_{\rm q}$ q-forms.

Let us consider a one-forms $\mathcal{A}$ and identify it with the gauge field on superspace,

$$\mathcal{A} = dz^M \mathcal{A}_M(z) = dz^M \mathcal{A}_M^{(r)}(z) i T^r, \tag{C.5}$$

where T^r denotes the generator of a gauge group.

The super transformation of superfields is defined by using differential operators Q_α, $\bar{\mathrm{Q}}_{\dot\alpha}$. The exterior derivative $d = dz^M \frac{\partial}{\partial z^M}$ does not commute with Q_α, $\bar{\mathrm{Q}}_{\dot\alpha}$. Namely, $d = dz^M \frac{\partial}{\partial z^M}$, which maps p-forms into (p+1)-forms, does not maps superfields into superfields. This tells us that the basis dz^M is not useful for supersymmetry. We will introduce a new basis in the following, in terms of which exterior derivatives map superfields into superfields.

The following differential operators D_α, $\bar{\mathrm{D}}_{\dot\alpha}$ and ∂_a commute or anticommute with Q_α, $\bar{\mathrm{Q}}_{\dot\alpha}$,

$$\begin{aligned} \mathrm{D}_a &\equiv \partial_a \,, \\ \mathrm{D}_\alpha &\equiv \frac{\partial}{\partial\theta^\alpha} + i(\sigma^a\bar\tau)_\alpha\partial_a \,, \\ \mathrm{D}_{\dot\alpha} &\equiv \bar{\mathrm{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\sigma^a)_{\dot\alpha}\partial_a \,. \end{aligned} \tag{C.6}$$

We will express these differential operators as D_A, where the capital letter A denotes the spacetime index a, the spinor indices α or $\dot\alpha$. The derivative $\frac{\partial}{\partial z^M}$ can be written in terms of D_A as follows:

$$\frac{\partial}{\partial z^M} = e_M{}^A \mathrm{D}_A \,, \tag{C.7}$$

where the matrix $e_M{}^A$ has the form:

$$e_M{}^A = \begin{array}{cc} & A \\ M & \begin{pmatrix} \delta_a{}^b & 0 & 0 \\ -i\,{}_\alpha(\sigma^b\bar\theta) & \delta_\alpha{}^\beta & 0 \\ -i\,{}_{\dot\alpha}(\theta\sigma^b) & 0 & -\delta_{\dot\alpha}{}^{\dot\beta} \end{pmatrix} \end{array} . \tag{C.8}$$

We define a matrix $e_A{}^M$ such that

$$\begin{aligned} e_A{}^M e_M{}^B &= \delta_A{}^B \,, \\ e_M{}^A e_A{}^N &= \delta_M{}^N \,. \end{aligned} \tag{C.9}$$

From this, $e_A{}^M$ has the form:

$$e_A{}^M = \begin{array}{cc} & M \\ A & \begin{pmatrix} \delta_a{}^b & 0 & 0 \\ i\,{}_\alpha(\sigma^b\bar\theta) & \delta_\alpha{}^\beta & 0 \\ -i\,{}_{\dot\alpha}(\theta\sigma^b) & 0 & -\delta_{\dot\alpha}{}^{\dot\beta} \end{pmatrix} \end{array} . \tag{C.10}$$

By using D_A and $e_M{}^A$, the exterior derivative $d = dz^M \frac{\partial}{\partial z^M}$ may be written as follows:

$$\begin{aligned} d = dz^M \frac{\partial}{\partial z^M} &= dz^M e_M{}^A \mathrm{D}_A \\ &= e^A \mathrm{D}_A \,, \end{aligned} \tag{C.11}$$

where we have introduced a new basis e^A defined by

$$e^A = dz^M e_M{}^A \,, \tag{C.12}$$

or, from(C.9),

$$dz^M = e^A e_A{}^M \,. \tag{C.13}$$

From now on we will call the index M Einstein index, while A flat index.

We may expressed p-forms with this basis e^A:

$$\begin{aligned} \Omega_{\rm p} &= dz^{M_1} \cdots dz^{M_{\rm p}} \Omega_{M_{\rm p} \cdots M_1}(z) \,, \\ &= e^{A_1} \cdots e^{A_{\rm p}} \Omega_{A_{\rm p} \cdots A_1}(z) \,. \end{aligned} \tag{C.14}$$

The relations between the components $\Omega_{M_{\rm p} \cdots M_1}$ and $\Omega_{A_{\rm p} \cdots A_1}$ are determined from the above equation and the definition of e^A. Since the matrix $e_M{}^A$ depends on θ, $\bar\theta$, the basis e^A does not commute with exterior derivatives,

$$\begin{aligned} [d, e^A] &= dz^M e^B \mathrm{D}_B e_M{}^A \\ &= e^C e_C{}^M e^B \mathrm{D}_B e_M{}^A \\ &= \begin{cases} 2i\sigma^a_{\alpha\dot\alpha} e^\alpha e^{\dot\alpha} & (A = a) \\ \ \ 0 & (\text{otherwise}) \end{cases} \\ &\equiv de^A \,. \end{aligned} \tag{C.15}$$

So in terms of e^A, $d\Omega_{\rm p}$ takes the form,

$$\begin{aligned} d\Omega &= e^{A_1} \cdots e^{A_{\rm p}} e^B \mathrm{D}_B \Omega_{A_{\rm p} \cdots A_1} \\ &\quad + e^{A_1} \cdots (de^{A_{\rm p}}) \Omega_{A_{\rm p} \cdots A_1} \\ &\quad + \cdots\cdots \\ &\quad + (de^{A_1}) \cdots e^{A_{\rm p}} \Omega_{A_{\rm p} \cdots A_1} \,. \end{aligned} \tag{C.16}$$

Note that de^A is expressed only with e^α, $e^{\dot\alpha}$ and does not depends on $z = (x, \theta, \bar\theta)$. Namely, in terms of e^A, each component of $d\Omega_{\rm p}$ is represented with D_A and the components $\Omega_{A_{\rm p} \cdots A_1}$ of $\Omega_{\rm p}$. This tells us that, in terms of e^A, if all components of $\Omega_{\rm p}$ are superfields, all components of $d\Omega_{\rm p}$ are also superfields. (Note that, in general, when $\Omega_{A_{\rm p} \cdots A_1}$ are superfields, $\Omega_{M_{\rm p} \cdots M_1}$ are not superfields.) From the above argument, we can conclude that the basis e^A is useful for supersymmetry.

Let us represent the one-forms $\mathcal{A}$ by using the basis e^A,

$$\begin{aligned} \mathcal{A} &= dz^M \mathcal{A}_M \\ &= e^A \mathcal{A}_A \,, \end{aligned} \tag{C.17}$$

and from this, the relations between $\mathcal{A}_M$ and $\mathcal{A}_A$ are

$$\begin{aligned} \mathcal{A}_{\underline{a}} &= \mathcal{A}_a \,, \\ \mathcal{A}_{\underline{\alpha}} &= \mathcal{A}_\alpha - i(\sigma^a \bar{\theta})_\alpha \mathcal{A}_a \,, \\ \mathcal{A}_{\underline{\dot{\alpha}}} &= -\mathcal{A}_{\dot{\alpha}} - i(\theta \sigma^a)_{\dot{\alpha}} \mathcal{A}_a \,, \end{aligned} \tag{C.18}$$

where $\underline{a}$, $\underline{\alpha}$ and $\underline{\dot{\alpha}}$ denote Einstein indices. Here we regard all the components $\mathcal{A}_A$ as superfields. The gauge transformation of the component $\mathcal{A}_A$ takes the form,

$$\mathcal{A}_A \longrightarrow \mathcal{A}'_A = -X^{-1} \mathrm{D}_A X + X^{-1} \mathcal{A}_A X. \tag{C.19}$$

The field strength $\mathcal{F}$ may be rewritten in terms of e^A as follows:

$$\begin{aligned} \mathcal{F} &= d\mathcal{A} + \mathcal{A}\mathcal{A} \\ &= e^A e^B \mathrm{D}_B \mathcal{A}_A + (de^A)\mathcal{A}_A \\ &\quad + e^A \mathcal{A}_A e^B \mathcal{A}_B \\ &= 2i\, e\sigma^a \bar{e} \mathcal{A}_a \\ &\quad + \frac{1}{2} e^A e^B \{ \mathrm{D}_B \mathcal{A}_A - (-)^{|A||B|} \mathrm{D}_A \mathcal{A}_B \\ &\qquad\qquad - \mathcal{A}_B \mathcal{A}_A + (-)^{|A||B|} \mathcal{A}_A \mathcal{A}_B \} \,, \end{aligned} \tag{C.20}$$

and components of $\mathcal{F}$ are

$$\begin{aligned} \mathcal{F}_{ab} &= \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a - [\mathcal{A}_a, \mathcal{A}_b] \,, \\ \mathcal{F}_{a\alpha} &= \partial_a \mathcal{A}_\alpha - \mathrm{D}_\alpha \mathcal{A}_a - [\mathcal{A}_a, \mathcal{A}_\alpha] \,, \\ \mathcal{F}_{a\dot{\alpha}} &= \partial_a \bar{\mathcal{A}}_{\dot{\alpha}} - \bar{\mathrm{D}}_{\dot{\alpha}} \mathcal{A}_a - [\mathcal{A}_a, \bar{\mathcal{A}}_{\dot{\alpha}}] \,, \\ \mathcal{F}_{\alpha\beta} &= \mathrm{D}_\alpha \mathcal{A}_\beta + \mathrm{D}_\beta \mathcal{A}_\alpha - \{\mathcal{A}_\alpha, \mathcal{A}_\beta\} \,, \\ \mathcal{F}_{\dot{\alpha}\dot{\beta}} &= \bar{\mathrm{D}}_{\dot{\alpha}} \bar{\mathcal{A}}_{\dot{\beta}} + \bar{\mathrm{D}}_{\dot{\beta}} \bar{\mathcal{A}}_{\dot{\alpha}} - \{\bar{\mathcal{A}}_{\dot{\alpha}}, \bar{\mathcal{A}}_{\dot{\beta}}\} \,, \\ \mathcal{F}_{\alpha\dot{\alpha}} &= \mathrm{D}_\alpha \bar{\mathcal{A}}_{\dot{\alpha}} + \bar{\mathrm{D}}_{\dot{\alpha}} \mathcal{A}_\alpha - \{\mathcal{A}_\alpha, \bar{\mathcal{A}}_{\dot{\alpha}}\} + 2i\sigma^a_{\alpha\dot{\alpha}} \mathcal{A}_a \,, \end{aligned} \tag{C.21}$$

where we define these by $\mathcal{F} = \frac{1}{2} e^A e^B \mathcal{F}_{BA}$. Each component of $\mathcal{F}$ represents a full superfield multiplet. These multiplets contain a large number of component fields. Most of the

component fields are superfluous and must be eliminated through constraint equations. The constraint equations which we should take are well-known as flatness condition:

$$\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\dot{\alpha}\dot{\beta}} = \mathcal{F}_{\alpha\dot{\alpha}} = 0\,. \tag{C.22}$$

Since our goal in this appendix is not to get the general solutions of (C.22), here we adopt a following solution [9, 10]

$$\begin{aligned}
\mathcal{A}_\alpha &= -\mathrm{e}^{-\mathrm{V}}\mathrm{D}_\alpha \mathrm{e}^{\mathrm{V}}\,, \\
\bar{\mathcal{A}}_{\dot{\alpha}} &= 0\,, \\
\mathcal{A}_a &= -\frac{i}{4}\bar{\sigma}_a^{\dot{\alpha}\alpha}\Big(\mathrm{D}_\alpha\bar{\mathcal{A}}_{\dot{\alpha}} + \bar{\mathrm{D}}_{\dot{\alpha}}\mathcal{A}_\alpha - \{\mathcal{A}_\alpha, \bar{\mathcal{A}}_{\dot{\alpha}}\}\Big) \\
&= \frac{i}{4}\bar{\sigma}_a^{\dot{\alpha}\alpha}\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{e}^{-\mathrm{V}}\mathrm{D}_\alpha \mathrm{e}^{\mathrm{V}}\,,
\end{aligned} \tag{C.23}$$

where $\mathrm{V} = \mathrm{V}^{(r)}T^r$ and e^{V} is an element of the gauge group generated by T^r. These equations trivially satisfy $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\dot{\alpha}\dot{\beta}} = 0$, because $\bar{\mathcal{A}}_{\dot{\alpha}} = 0$ and $\mathcal{A}_\alpha$ has the form of pure gauge type. $\mathcal{A}_a$ is determined to satisfy $\mathcal{F}_{\alpha\dot{\alpha}} = 0$ for the given $\bar{\mathcal{A}}_{\dot{\alpha}}$, $\mathcal{A}_\alpha$.

We will assume that V is a real superfield;$\mathrm{V} = \mathrm{V}^\dagger$. V has the following form:

$$\begin{aligned}
\mathrm{V} = {} & \mathrm{C}(x) + i\theta\chi(x) + i\bar{\theta}\bar{\chi}(x) \\
& + \frac{i}{2}\theta\theta[\mathrm{M}(x) + i\mathrm{N}(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[\mathrm{M}(x) - i\mathrm{N}(x)] \\
& - \theta\sigma^a\bar{\theta}v_a(x) + i\theta\theta\bar{\theta}[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^a\partial_a\chi(x)] \\
& - i\bar{\theta}\bar{\theta}\theta[\lambda(x) + \frac{i}{2}\sigma^a\partial_a\bar{\chi}(x)] \\
& + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[\mathrm{D}(x) + \frac{1}{2}\Box\mathrm{C}(x)]\,,
\end{aligned} \tag{C.24}$$

where the component fields $\mathrm{C}, \mathrm{D}, \mathrm{M}, \mathrm{N}$ and v_a are all real from the reality condition of V. V is called vector superfield and is a fundamental superfield in supersymmetric gauge theories. Since we have kept explicitly supergauge covariance in these arguments, the solutions which are given by exchanging e^{V} for $e^{\mathrm{V}}X$ in (C.23) are also satisfy the flatness conditions. V has another ambiguity. Even if we exchange e^{V} for $e^{\Lambda^\dagger}e^{\mathrm{V}}$ where $\Lambda^\dagger$ satisfies $\mathrm{D}_\alpha\Lambda^\dagger = 0$, the equations (C.23) does not change. Combining this ambiguity with the gauge transformation in superspace, the transformation of V takes the form $e^{\mathrm{V}} \to e^{\mathrm{V}'} = e^{\Lambda^\dagger}e^{\mathrm{V}}X$. Since V' should also be real, X must be equal to e^Λ. Therefore V transforms as follows:

$$\begin{aligned}
e^{\mathrm{V}} \;\longrightarrow\; & e^{\mathrm{V}'} = e^{\Lambda^\dagger}e^{\mathrm{V}}e^{\Lambda}\,, \\
& \bar{\mathrm{D}}_{\dot{\alpha}}\Lambda = 0\,, \\
& \mathrm{D}_\alpha\Lambda^\dagger = 0\,.
\end{aligned} \tag{C.25}$$

We will call this transformation the restricted super gauge transformation in this paper because the element X of super gauge must satisfy $X = \mathrm{e}^{\Lambda}$, $\bar{\mathrm{D}}_{\dot\alpha}\Lambda = 0$ in the above transformation.(note that, in the case $X = \mathrm{e}^{\Lambda}$, $\mathcal{A}_{\dot\alpha}$ remains zero, $\mathcal{A}_{\dot\alpha} \to \mathcal{A}'_{\dot\alpha} = -\mathrm{e}^{-\Lambda}\mathrm{D}_{\dot\alpha}\mathrm{e}^{\Lambda} = 0$.)

Let us define a new superfield $\mathcal{W}_\alpha$,

$$\begin{aligned}\mathcal{W}_\alpha &= \frac{i}{2}\sigma^a_{\alpha\dot\alpha}\bar{\mathrm{D}}^{\dot\alpha}\mathcal{A}_a \\ &= -\frac{1}{4}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{e}^{-\mathrm{V}}\mathrm{D}_\alpha\mathrm{e}^{\mathrm{V}}\,. \end{aligned}\tag{C.26}$$

In terms of $(y, \theta, \bar\theta)$, $\bar{\mathrm{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}$. From this we see that

$$\bar{\mathrm{D}}_{\dot\alpha}\bar{\mathrm{D}}\bar{\mathrm{D}} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta^{\dot\beta}}\frac{\partial}{\partial\bar\theta_{\dot\beta}} = 0\,,$$

and, furthermore,

$$\bar{\mathrm{D}}_{\dot\alpha}\mathcal{W}_\alpha = 0\,. \tag{C.27}$$

Namely $\mathcal{W}_\alpha$ is a chiral superfield. The transformation of $\mathcal{W}_\alpha$ which follows (C.25) takes the form,

$$\mathcal{W}_\alpha \longrightarrow \mathcal{W}'_\alpha = e^{-\Lambda}\mathcal{W}_\alpha e^{\Lambda}\,. \tag{C.28}$$

References

[1] Yu.M. Makeenko and A.A. Migdal, *Phys. Lett.* **B88** (1979) 135.

[2] A.A. Migdal, *Phys. Rep.* **102** (1983) 199.

[3] A.M. Polyakov, *"Contemporary Concepts in Physics Volume 3; Gauge Fields and Strings", harwood academic publishers.*

[4] H.Itoyama and H.Takashino, *Phys. lett.* **B381** (1996)163.

[5] S. J. Gates, *Phys. Rev.* **D16** (1977) 1727.

[6] S. Marculescu, *Nucl. Phys.* **B213** (1983) 523.

[7] S. J. Gates, M. Grisaru, M. Rocek and W. Siegel, *"Superspace" Benjamin/Cummings* (1983).

[8] J. Wess and J. Bagger, *"Supersymmetry and Supergravity"*, *Princeton University Press* (1992) 2nd edition.

[9] M. F. Sohnius, *Phys. Rep.* **128** (1985) 39.

[10] J-L. Gervais, M.T. Jaekel and A. Neveu, *Nucl. Phys.* **B155** (1979) 75.

[11] Yu.M. Makeenko and P.B. Medvedev, *Nucl. Phys.* **B193** (1981) 444.

[12] J.Ishida, A.Hosoya, *Prog. Theor. Phys.* **62** (1979)544.

[13] J. L. Gervais, A. Neveu, *Nucl. Phys.* **B163** (1980)189 .

[14] I. Aref'eva, *Phys. Lett.* **93B** (1980)347.

[15] V. S. Dotsenko, S. N. Vergeles, *Nucl. Phys.* **B169** (1980)527.

[16] N. S. Cragie, H. Dorn, *Nucl. Phys.* **B185** (1981)204.

[17] S. Aoyama, *Nucl. Phys.* **B194** (1982)513.

[18] N. Seiberg and E. Witten, *Nucl. Phys.* **B426** (1994)19: *Nucl. Phys.* **B431** (1994)484. See K. Intriligator and N. Seiberg *Nucl. Phys. Proc. Suppl.* **45BC** (1996)1,hep-th 9509066 for more references.

[19] A.Gorsky,I.Krichever,A.Marshakov,A.Mironov and A.Morozov *Phys. Lett.* **355B** (1995)466, hep-th/9505035; E.Martinec and N.Warner, *Nucl. Phys.* **B459** (1996)97,hep-th/9509161; T.Nakatsu and K.Takasaki, *Mod. Phys. Lett.* **A11** (1996)157, hep-th/9509162; R.Donagi and E.Witten, *Nucl. Phys.* **B460** (1996)299, hep-th/9510101; T.Eguchi and S.K.Yang, hep-th/9510183; E.Martinec, *Phys. Lett.* **B 367** (1996)91, hep-th/9510204; H.Itoyama and A.Morozov, *Nucl. Phys.* **B477** (1996)855,hep-th/9511126; hep-th/9512161,*Nucl. Phys.* **B488** [PM] to appear.